Available online at www.sciencedirect.com
SCIENCE DIRECT

Physica A 343 (2004) 247–262

PHYSICA A

www.elsevier.com/locate/physa

The presence of an invariant manifold in the dissipative nonlinear four-wave coupling

J.C.P. Coninck, S.R. Lopes*, R.L. Viana

Departamento de Física, Universidade Federal do Paraná, Curitiba, Paraná, 81531-990 Brazil

Received 8 June 2004
Available online 29 july 2004

Abstract

We examine the existence of an invariant manifold in the phase space structure presented by a dissipative four-wave coupling. We show that, once in such manifold, the system preserves two quantities as time evolves: a quantity related to the energy of the system and another, related to the exchange of energy between the waves. We observe that the presence of this manifold allows the system to manifest stable solutions for a specific interval of the dissipation parameters. Out of this range the system brings up only uncoupled states (no nonlinear saturated states). The dissipation parameters affect only the transversal dynamics to the manifold, changing how quickly the invariant manifold is reached. In the invariant manifold, the system exhibits a large number of coexisting periodic attractors—limit cycles (coupled states), with an interwoven structure of basins of attraction. The time series of conserved quantities and the properties of the Lyapunov spectra are used to imply the existence of a lower-dimensional invariant manifold. Since the dynamics in such invariant manifold preserves an energy-like function, we identify it as being conservative.

PACS: 52.35.Mw; 05.45.−a

Keywords: Wave coupling; Invariant manifold; Dissipative systems

*Corresponding author.
E-mail address: lopes@fisica.ufpr.br (S.R. Lopes).

0378-4371/$ - see front matter
doi:10.1016/j.physa.2004.07.008

1. Introduction

It is a common feature of dissipative dynamic system to exhibit a great variety of stationary asymptotic behaviors such as periodic, quasiperiodic and chaotic attractors. Qualitative changes of those attractors can be investigated by a bifurcation analysis, such that we can observe how the dynamics of the system changes as a parameter is varied. Much work has focused on low-dimensional systems where only one or two attractors are present. In spite of that, many systems in Nature are more appropriately modeled by dynamical systems having multiple coexisting attractors. In such systems, the dynamics will be much more complex since interactions among the various attractors and their basins should be taken into account [1,2]. The complexity of the so-called multistable system was first investigated using prototype models [3,4] for example, the periodically kicked rotor. For this case, in the limit of strong damping, the system reduces to a one-dimensional circle map with zero rotation number exhibiting the Feigenbaum scenario to chaos. On the other hand, without damping we obtain the area-preserving standard map [5]. But, when a small amount of dissipation is applied in the Hamiltonian case of the rotor map, it will possess a finite number of stable periodic orbits, such that a study of the appearance and disappearance of attractors can be performed [1]. In general, multistability is a common feature in Hamiltonian system with weakly dissipative effects.

Here we investigate the multistability and the presence of a low-dimensional invariant manifold that arises from the four-wave interaction in a continuous media. We show that the presence of such a manifold enables the system to exhibit stable coupled states for a relatively large interval of the dissipation parameters.

It is well known that resonant nonlinear wave interactions are of great interest in many branches of physics. Such kind of interactions are common processes in plasma physics [6–9], playing an important role in both space and laboratory plasma processes, such as filamentation and anomalous absorption of beams [10], generation of auroral and solar emissions [11] and modulation of solar wind Langmuir waves [12]. Resonant interactions can be found in nonlinear optics as well [13,14].

In particular, four-wave interaction is of fundamental importance in the nonlinear wave theory. In this interaction process two-wave-triplets sharing two common modes are coupled such that the modes involved keep exchanging energy with time. This system was first studied by Sugihara [15] and Karplyuk et al. [16], who have derived some particular solutions for the conservative coupling, considering the case of perfect matching conditions. For the conservative regime and perfect matching condition, the integrability of the four-wave problem was proved by Romeiras [17]. Concerning the conservative regime and allowing imperfect resonant conditions, this type of interaction has been shown to produce amplitude equation allowing for temporal chaos [8]. Little work, however, has been done on solutions for the dissipative case.

Recently, we have shown that, for a select group of parameters (dissipation and mismatches), the dissipative four-wave interaction presents a rich dynamical behavior revealing that the system exhibits a growing number of attractors as the frequency mismatch of both triplets turns to be the same [18]. Such equality of the frequency mismatch parameters has made it possible for us to define a symmetry parameter, which measures the total mismatch between triplets (the difference between both frequency mismatches). We have shown that, for the case of equal total mismatch, the number of attractors goes to infinity. In this situation the system presents a conservative invariant manifold, where the dynamics preserve a function related to the system energy, or, in other words, the system presents stationary states where an energy related constant is conserved as the system evolves in time.

Here we focus on the effect of the dissipative parameters, we show that these parameters affect only the transversal dynamics to the manifold, changing how quickly the invariant manifold is reached but they do not change the conservative nature of the manifold. As we discuss later, such behavior seems to be a result of the symmetry of the system only. We call a system as being symmetric when both triplets have the same frequency mismatch. In such a case, the energy of a pump wave is equally distributed between both triplets through an idle wave that has a constant amplitude in time.

We point out that the existence of such multistable states does not depend on the magnitude of the mismatches or dissipation parameters for a relatively large intervals of their values.

The paper is organized as follows: In Section 2, we present the model of the four-wave resonant coupling. In Section 3 the symmetric coupled system is discussed. In Section 4 we argue about the presence of a conservative invariant manifold. Finally in Section 5 we present our conclusions.

2. The four-wave interaction model

Here we consider the resonant four-wave parametric interaction involving the coupling of two-wave triplet interaction (the three-wave problem) with resonant frequency condition given by

$$\omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_3} + \omega_{\mathbf{k}_2} + \delta_1 \, , \tag{1}$$

$$\omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_4} - \omega_{\mathbf{k}_2} - \delta_2 \, , \tag{2}$$

where $\omega_{\mathbf{k}_j}$, $(j = 1, 2, 3, 4)$ are the fast frequencies and $\delta_{1,2}$ stand for small linear frequency mismatches for each of the two-wave-triplets $(1, 2, 3$ and $1, 2, 4$, respectively), a usual effect in wave–wave interaction. A schematic representation of this coupling is depicted in Fig. 1. As we will see later in this paper, the frequency mismatch is essential in the study of the multistability of the model. From Eqs. (1) and (2), and for the case of $\delta_1 = \delta_2$, we have $2\omega_1 = \omega_3 + \omega_4$, which has the same result we get for the case without mismatches.

The resonant condition for the wave-vectors is written as

$$\mathbf{k}_1 = \mathbf{k}_3 + \mathbf{k}_2 \, , \tag{3}$$

$$\mathbf{k}_1 = \mathbf{k}_4 - \mathbf{k}_2 \tag{4}$$

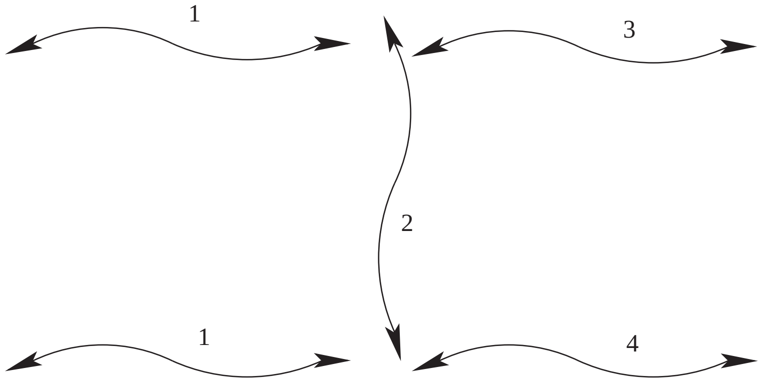

Fig. 1. Scheme of the four-wave interaction.

and we define the total mismatch parameter—the symmetry breaking parameter as

$$\Delta = \delta_1 - \delta_2 \, . \tag{5}$$

Wave vector mismatches can also be incorporated into the theory. However, as frequency and wave vector mismatches are formally equivalent in our case, we focus attention on the former.

Following the model revised in a series of recent papers [8,18,19], the dimensionless amplitude equations governing the slow modulation dynamics of the temporal four-wave interaction can be cast into the form

$$\frac{dA_1}{dt} = A_2 A_3 - r A_2^* A_4 + v_1 A_1 \, , \tag{6}$$

$$\frac{dA_2}{dt} = -A_1 A_3^* - r A_1^* A_4 + v_2 A_2 \, , \tag{7}$$

$$\frac{dA_3}{dt} = -A_1 A_2^* + i\delta_1 A_3 + v_3 A_3 \, , \tag{8}$$

$$\frac{dA_4}{dt} = r A_1 A_2 - i\delta_2 A_4 + v_4 A_4 \, , \tag{9}$$

where A_j, ($j = 1, 2, 3, 4$), are the complex amplitudes of the four fields and v_j are the linear growing/damping rates for each wave. Here we assume that $v_1 > 0$ and $v_{2,3,4} < 0$. r is a variable strength factor measuring the intensity of the triplet–triplet coupling, and will be set equal to 1 through this paper [8,20].

Since each wave field is characterized by two real parameters (amplitude F_i and phase ϕ_i, $A_i = F_i^{1/2} \exp \phi_1$), they are out of eight variables in this system, such that its phase space should be 8-dimensional, however, it is possible to show that the phase space can be reduced to a 6-dimensional space by using the phase conjugacies

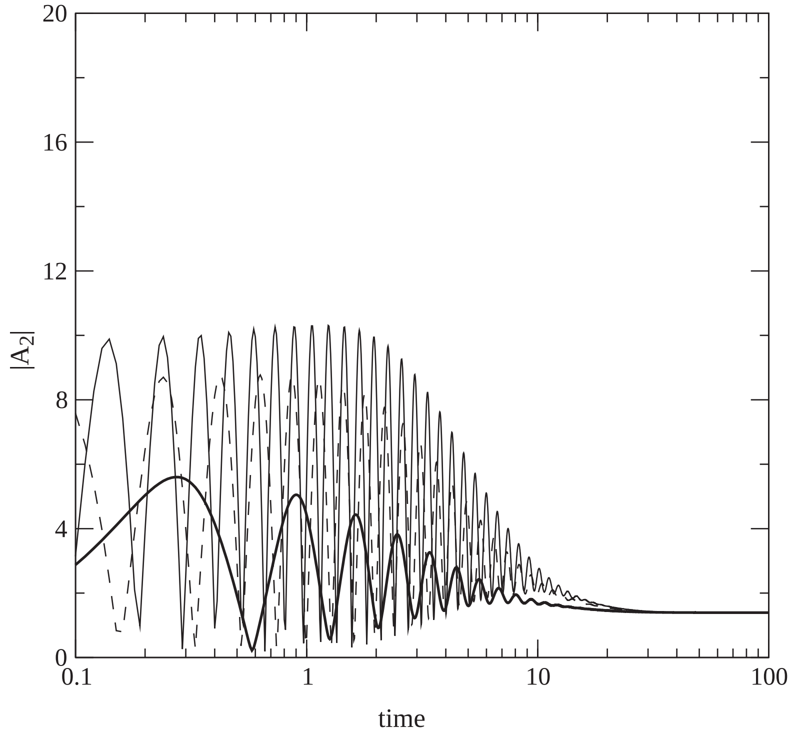

Fig. 2. Time evolution of wave 2 amplitude for the resonant four-wave parametric interaction for $\Delta = 0$ and $v_{2,3,4} = -0.8$.

given by the Stokes and Anti-Stokes modes [8]

$$\phi_+ = \phi_1 + \phi_2 - \phi_4 , \tag{10}$$

$$\phi_- = \phi_1 - \phi_2 - \phi_3 . \tag{11}$$

For all cases studied here, we set the dissipation coefficients v_i to have nonzero values, such that the system will be dissipative. The change of phase space volume with time, which is the divergence of the vector field (6)–(9),

$$\mathscr{J} = \sum_{i=1}^{4} \frac{\partial \dot{A}_i}{\partial A_i} = 2 \sum_{i=1}^{4} v_i , \tag{12}$$

is a constant value. Here we set $v_1 = 1$ and the values of $v_{2,3,4}$ are chosen such that the summation in Eq. (12) is always negative, indicating that the system is globally dissipative, i.e., the volumes shrink with time at the same rate everywhere in the available regions of the phase space.

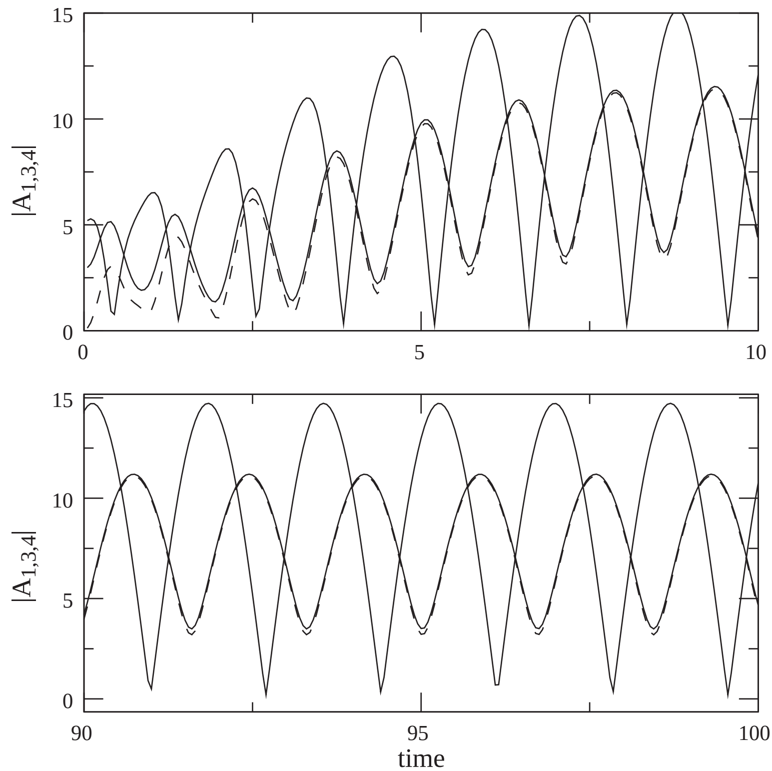

Fig. 3. Time evolution of waves 1, 3 and 4 for the resonant four-wave parametric interaction for $\Delta = 0$ and $v_{2,3,4} = -0.8$.

3. The symmetric interaction case

In order to investigate the effect of the perfect symmetry between both wave triplets in the system, we set in this section $\Delta = 0$ and $r = 1$. In the four-wave interaction analyzed here, the energy is transferred from/to wave 1 to/from wave 2, 3 and 4. In a general case all waves maintain oscillating and the energy keeps exchanging between them. But in the particular case $\Delta = 0$ and $r = 1$, and for a select interval of the dissipation parameters, we call it the symmetric case, the wave 2 works as a catalyst for the process. In fact, in Fig. 2 we plot the temporal behavior of wave 2 for three randomly chosen initial conditions. As far as the system approaches the invariant manifold (from time $t = 0$ to 50 approximately in the figure), the temporal behavior of the amplitude of the wave 2 tends to a constant value, while waves 1, 3 and 4 stay oscillating as can be seen from Fig. 3, where we plot the transient time (upper panel) and the asymptotic behavior of the time series for the wave amplitudes (down panel). Such catalytic role for wave 2 can also be observed for the integrable (without mismatches) Hamiltonian case of the interaction, for details see Ref. [21].

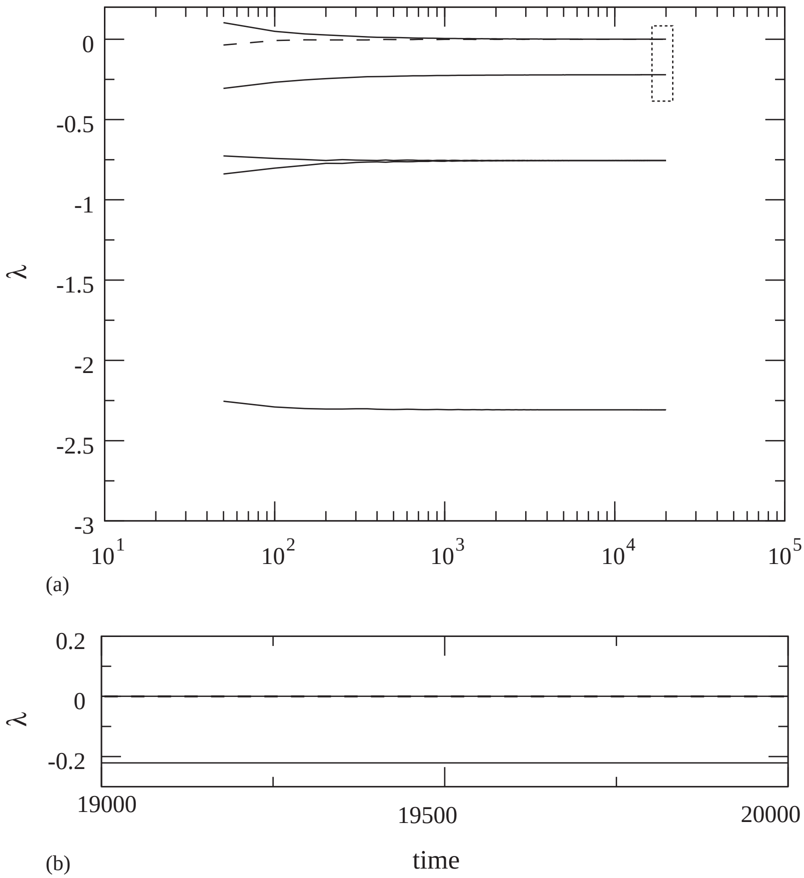

Fig. 4. (a) Time evolution of the Lyapunov exponents for the resonant four-wave parametric interaction for $\Delta = 0$, $v_{2,3,4} = -0.8$. (b) Magnification of (a).

For this symmetric case, we can observe that the dynamics is strongly dissipative along four out of the six phase-space dimensions, which directs trajectories asymptotically to a two-dimensional manifold $\mathscr{M}$. A numerical evidence for this assertion is depicted in Fig. 4, where we plot the time variation of the numerical value of the Lyapunov spectrum. The two-dimensional character of $\mathscr{M}$ is inferred from the Lyapunov spectrum, where all the six Lyapunov exponents are depicted. They are all strictly non-positive, indicating the absence of chaotic behavior. While four exponents are negative, we have two exponents equal to zero, indicating that the trajectories are pushed to this manifold along the remaining four directions in phase space.

The dynamics in this manifold can be parameterized by any set of suitable coordinates. We choose to work with the field amplitudes $|A_i|$. Fig. 5

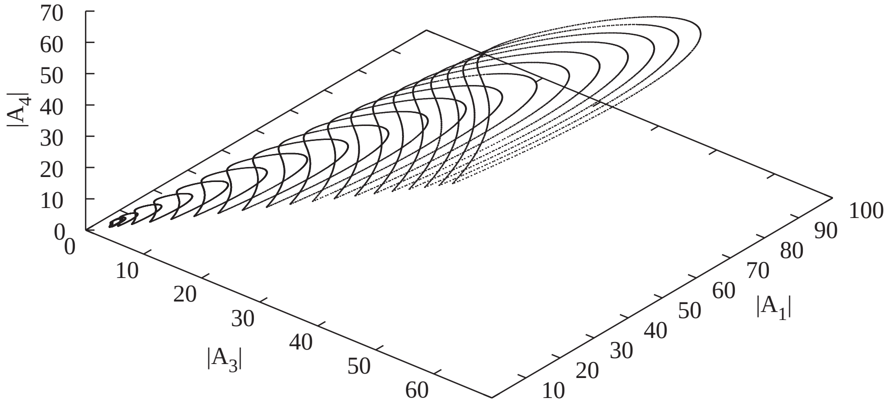

Fig. 5. Multiple coexisting period-1 attractors in the $|A_4| \times |A_3| \times |A_1|$ projection of the phase space for $\Delta = 0$ and $v_{1,2,3} = -0.8$.

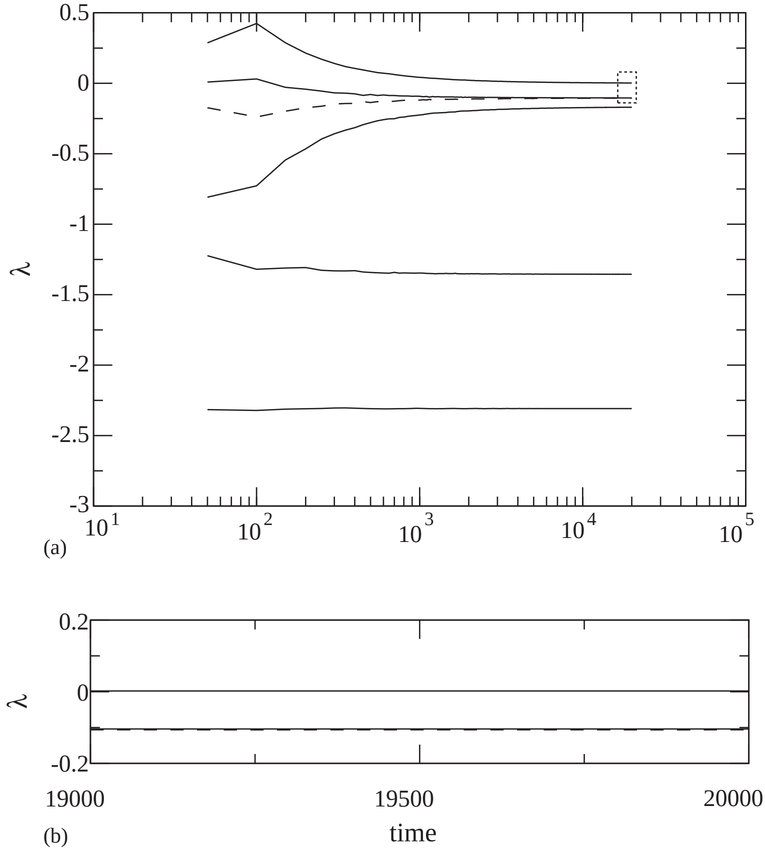

Fig. 6. (a) Time evolution of the Lyapunov exponents for the resonant four-wave parametric interaction $\Delta = 0.01$ and $v_{2,3,4} = -0.8$. (b) Magnification of (a).

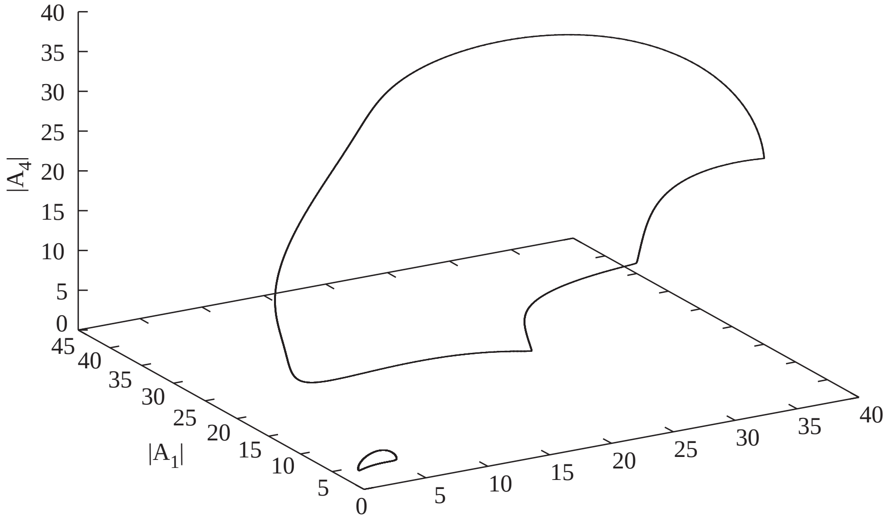

Fig. 7. Limit-cycle attractors in the $|A_4| \times |A_1| \times |A_3|$ projection of the phase space for $\Delta = 0.01$ and $v_{2,3,4} = -0.8$.

shows a projection of the phase space trajectories on the $|A_1| - |A_3| - |A_4|$ hyper-plane, in which a variety of different limit cycles attractors are shown. Here, each limit cycle was obtained using random initial condition. In fact, we have numerical evidences that the number of different limit cycles can be infinity since for each initial condition given to the system, it evolves for a different amplitude limit cycle. As can be observed, all limit cycles are contained in a same two-dimensional subspace (the invariant manifold) in the phase space.

For the $\Delta \neq 0$ case, such invariant manifold does not exist at all, as one can see from the variation of the Lyapunov spectrum shown in Fig. 6. In the figure, we observe just one vanishing Lyapunov exponent, the one along the trajectory. In this case the system does not exhibit the invariant manifold, and the asymptotic behavior of the system is governed by two limit cycles as observed in Fig. 7, where we plot the two asymptotic states presented by the system for the case $\Delta = 0.01$. For this case, the temporal behavior of wave two is no longer constant but oscillates with small amplitude.

4. Conservative dynamics in the invariant manifold

As long as the system settles down in the invariant manifold its dynamics preserves same quantities. The conservative character of the invariant manifold can be inferred

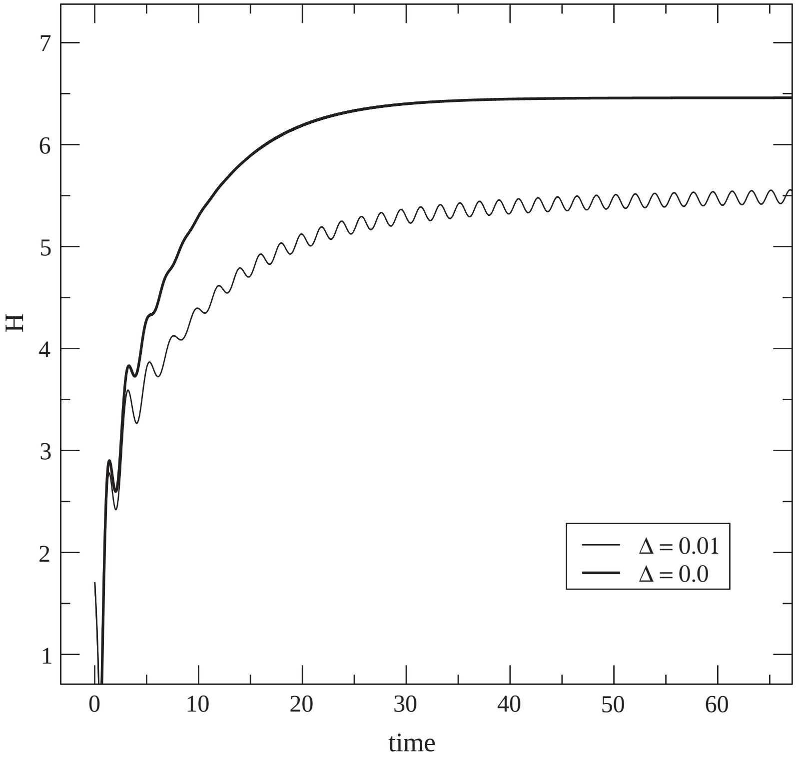

Fig. 8. Time evolution of the energy function H for $\Delta = 0$ and 0.01.

by observing the temporal behavior of the energy-like function $H(t)$ of the system

$$H(A_i) = A_1 A_2^* A_3^* - A_2 A_3 A_1^* + r(A_1^* A_2^* A_4 - A_1 A_2 A_4^*) + i(\delta_4 |A_4|^2 - \delta_3 |A_3|^2)\,. \tag{13}$$

In the conservative case this quantity is the Hamiltonian of the system and it is rigorously conserved but, in the dissipative case, it is no longer constant in general. But for the special case of $\Delta = 0$, and for some intervals of values of $v_{2,3,4}$, the dynamics of the system in the invariant manifold will preserve the value of this quantity, such that the invariant manifold can be regarded as a conservative one, since it preserves a quantity related to the energy of the system. The temporal behavior of the energy-like function $H(t)$ is plotted in Fig. 8, where we show that $H(t)$ oscillates for some time (while the system is not yet in the invariant manifold) until it saturates at a constant value for the case ($\Delta = 0$). As H tends to a constant value, this suggests that trajectories approach a phase space region where the dynamics is conservative. For the case of $\Delta \neq 0$, $H(t)$ does not settle down at any

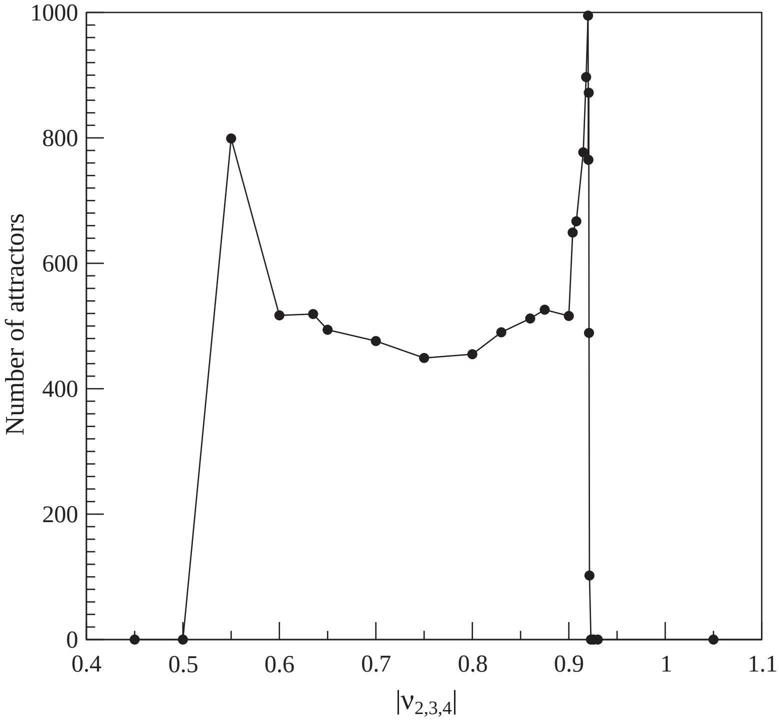

Fig. 9. Interval of the dissipation parameters where the system displays a invariant manifold for $\Delta = 0$.

constant value, in fact, it keeps oscillating as time evolves. This behavior can be observed, for a particular value of $\Delta = 0.01$, in Fig. 8. In this case the system does not present the invariant conservative manifold.

As mention before in this paper, for the symmetric case ($\Delta = 0.0$) the system presents a great number of periodic attractors. In Fig. 9 we plot the number which these attractors exhibited by the system as a function of the dissipation parameters $v_{2,3,4}$. As we can observe, the invariant manifold exhibited by the system, exists for a relatively large range of the dissipation parameters $v_{2,3,4}$. In fact the number plotted is just a lower bound for the real number of attractors. The conservative nature of the invariant manifold enables the system to present an infinite number of limit cycles. Nevertheless, as our results are computed numerically just a lower bound can be acquired, since many basin boundaries are too small to be resolved due to finite graphical resolution procedures.

As we have just pointed out, for the case where the system asymptotic dynamics is confined in a conservative manifold, the system presents an infinite number of stationary limit cycles as its final state. In the face of such result, it is important to see how sensitive our method is, to compute different final states. In order to see that, we

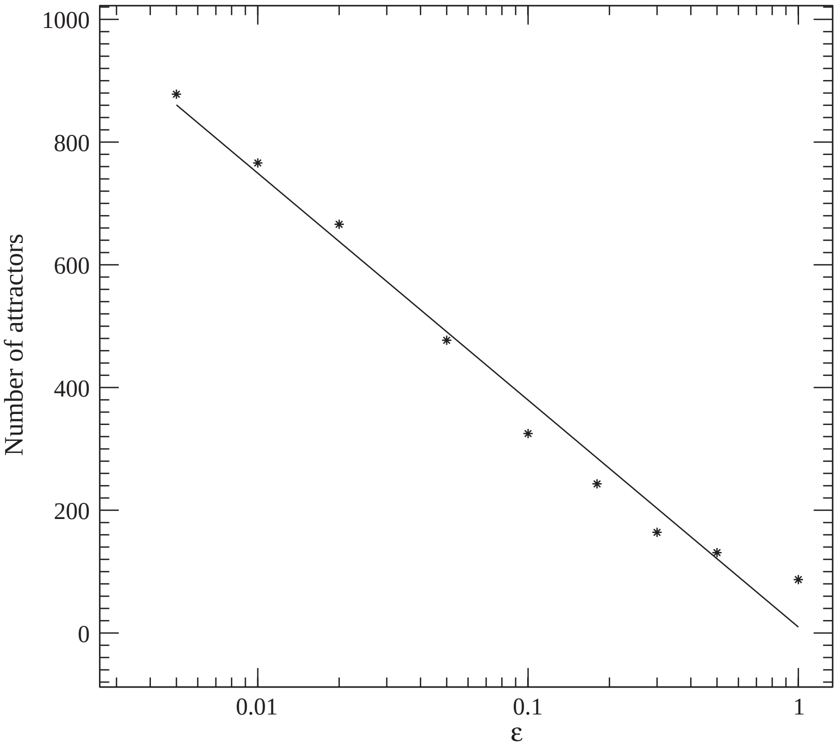

Fig. 10. Number of attractors as a function of the tolerance admitted by our numerical method to identify an attractor.

have simulated 45 000 initial conditions for an interval of ε-tolerance, and have established for what limit cycles the system is attracted to. To do so we specify a tolerance in the cycle amplitude ε. Such tolerance tells us that all cycles found in the interval $X \mp \varepsilon$ will be computed as just one limit cycle. In Fig. 10, we plot the dependence in the number of attractors found for the system as a function of the error in determining the limit-cycle amplitude. The number of attractors obeys a logarithmic dependence with the tolerance. Such results point out that the system presents an infinity number of stable limit cycles for the case $\Delta = 0.0$.

As we should expect, the asymptotic energy value of the system tends to be bigger when the dissipation parameters assume smaller values. In order to see this dependence, we plot in Fig. 11 the asymptotic value assumed by $H(t)$ (the energy-like function) as a function of the dissipation parameters $v_{2,3,4}$. It changes in a power law fashion with the dissipation parameters.

The transient time (the time until the system settles down in the invariant manifold) tends to be smaller for bigger values of the dissipation parameters. This can be understood since the stronger the dissipation the quicker the system goes to

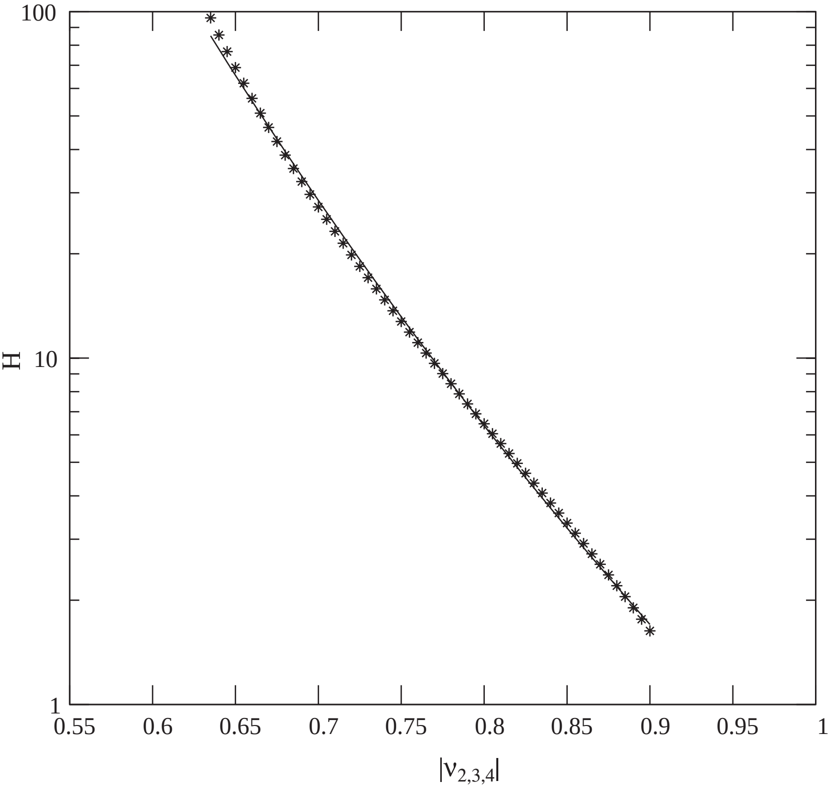

Fig. 11. Asymptotic energy function ($H(t)$) as a function of the dissipative parameters $v_{2,3,4}$ for $\Delta = 0.0$.

the invariant manifold through the four dissipative directions perpendicular to this manifold. Such a statement can be better understood when we plot the time until the value of $H(t)$ settles down into a constant value as can be observed in Fig. 12. Nevertheless, the change of the values of the dissipation parameters does not change the presence of the invariant manifold, that exist for the interval of $0.5 < v_{2,3,4} < 0.9$ approximately. In Fig. 13, we plot the bifurcation diagram as a function of the dissipation parameters ($v_{2,3,4}$). In the figure, we observe that for $v_{2,3,4} < -0.92$ and $v_{2,3,4} > -0.5$ no stationary states can be found. Just in the interval $-0.92 < v_{2,3,4} < -0.5$ saturated states exist. This interval is exactly where we find the presence of the invariant manifold.

5. Conclusions

In its conservative regime, the nonlinear interaction of four-waves, is integrable provided there is perfect frequency matching between the waves. Such a result was

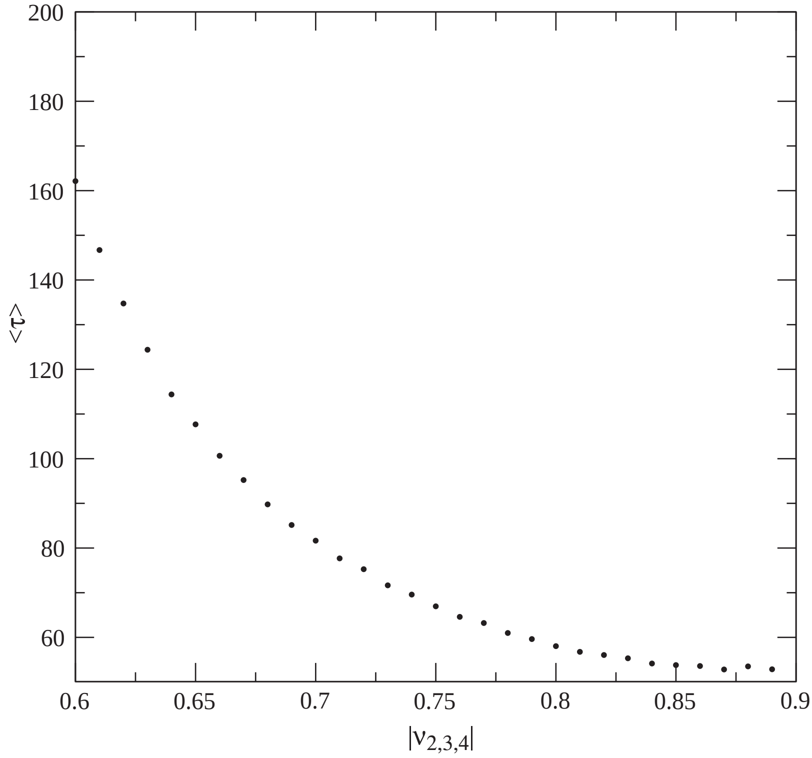

Fig. 12. Average transient time over 500 initial conditions for the system settles down in the invariant manifold as a function of the dissipations parameters $\nu_{2,3,4}$ for $\Delta = 0$.

formally proved by Romeiras [17]. The introduction of a small nonzero frequency mismatch breaks down the system integrability and opens the possibility of chaotic dynamics [8]. In this paper we have shown that the addition of small dissipative terms to the otherwise conservative model leads to a complex phase space structure, whose main feature is the presence of a invariant manifold where the dynamics preserves an energy-like constant. We have exhibited that for a wide interval of the dissipation parameters and for the symmetric ($\Delta = 0$) case the system trajectories, albeit belonging to a high-dimensional phase space, are asymptotically convergent to a two-dimensional invariant manifold in which the dynamics is conservative in the sense that it preserves an energy constant. We have numerically observed that, as long as the invariant manifold exists, the dynamics present a nonlinear saturated state (a coupled state for the wave system). Out of the interval range for the dissipation parameter where the invariant manifold is stable, the system shows just uncoupled states or in other words, no nonlinear saturation can be found. In face if we argue that the stability of the invariant manifold brings up nonlinear saturation to the dynamics of the four-wave coupling.

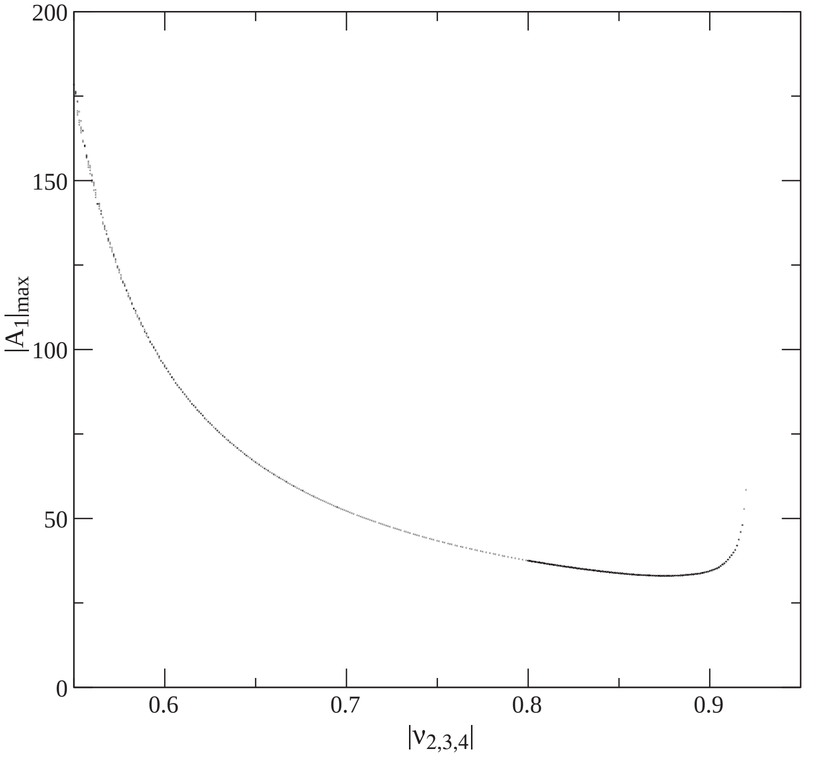

Fig. 13. Bifurcation diagram of the system as a function of $\nu_{2,3,4}$ for $\Delta = 0$.

Acknowledgements

This work was partially supported by CNPq and CAPES.

References

[1] U. Feudel, C. Grebogi, L. Poon, J. Yorke, Chaos, Solitons Fractals 9 (1998) 171.
[2] L. Poon, C. Grebogi, Phys. Rev. Lett. 55 (1995) 908.
[3] S.E. Newhouse, Topology 13 (1974) 9.
[4] H.E. Nusse, J.A. Yorke, Physica D 57 (1992) 39.
[5] B. Chirikov, Phys. Rep. 52 (1979) 263.
[6] J. Weiland, H. Wilhelmsson, Coherent Non-Linear Interactions of Waves in Plasmas, Pergamon Press, New York, 1977.
[7] A.C.-L. Chian, S.R. Lopes, M.V. Alves, Astron. Astrophys. 290 (1994) L13.
[8] R. Pakter, S.R. Lopes, R.L. Viana, Physica D 110 (1997) 277.
[9] S.R. Lopes, F.B. Rizzato, Physica D 117 (1998) 13.
[10] A.C.-L. Chian, F.B. Rizzato, J. Plasma Phys. 51 (1994) 61.
[11] S.R. Lopes, Astron. Astrophys. 305 (1996) 669.

[12] A.C.-L. Chian, J.R. Abalde, M.V. Alves, S.R. Lopes, Solar Phys. 173 (1997) 199.
[13] Y.R. Shen, The Principle of Nonlinear Optics, Wiley, New York, 1984.
[14] A. Yariv, Quantum Electronics, Wiley, New York, 1989.
[15] R. Sugihara, Phys. Fluids D 11 (1968) 178.
[16] K.S. Karplyuk, V.N. Oraevskii, V.P. Pavlenko, Plasma Phys. 15 (1973) 113.
[17] F.J. Romeiras, Phys. Lett. A 93 (1983) 227.
[18] J.C.P. Coninck, S.R. Lopes, R.L. Viana, Phys. Rev. E (2004), submitted for publication.
[19] S.R. Lopes, F.B. Rizzato, Phys. Rev. E 60 (1999) 5375.
[20] A.C.-L. Chian, S.R. Lopes, J.R. Abalde, Physica D 99 (1996) 269.
[21] D. Walters, G.J. Lewak, J. Plasma Phys. 18 (1977) 525–536.