23 July 2001

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 286 (2001) 134–140

www.elsevier.com/locate/pla

Lyapunov exponents of a lattice of chaotic maps with a power-law coupling

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Received 2 January 2001; received in revised form 12 June 2001; accepted 13 June 2001
Communicated by A.P. Fordy

Abstract

We consider a lattice of coupled chaotic logistic maps in which the coupling strength between sites decreases with the lattice distance in a power-law fashion, making possible to pass continuously from a local to a global coupling. The corresponding Lyapunov spectra are described by means of the maximal exponent and the average of the positive exponents, which are analyzed in terms of the coupling properties. Our results are compared with spatio-temporal patterns known for global and local couplings.

Lattice dynamical systems have been extensively studied as models of spatio-temporal complexity, due to the interplay between time and spatial degrees of freedom. Many phenomena observed in fluids and plasmas, like soliton propagation, traveling waves and turbulence — only to name a few — are also present in lattice models, that are still able to retain some features of complex natural phenomena [1]. One of the most widely investigated types of spatially extended systems is the coupled map lattice (CML), in which both space and time are discrete variables, but with a continuous state variable [2]. A CML is basically composed by a local dynamical unit that undergoes discrete temporal evolution, interacting with other units through a given coupling prescription.

The type of the coupling should reflect the properties of the interaction between model units. For example, if the coupled map lattice purports to describe a chain of spheres or pendula linked by soft springs, it would be natural to use a nearest-neighbor coupling [3,4]. On the other hand, in a coupled map lattice model of the brain function, each unit representing a neuron, the coupling would have to include a bunch of neighbors, reflecting the clustering nature of the synaptic connections [5]. This calls for a globally coupled model [6]. In an extreme situation, the coupling is such that each unit interacts with a kind of mean field produced by every single unit in the lattice [6,7]. The dynamics of globally coupled map lattices has been extensively studied in recent years. One of the outstanding features of these systems is the presence, in the turbulent regime, of various periodic cluster attractor states, even though the coupling is very small when compared with the map nonlinearity. In particular, for coupled logistic maps there are a period-three cluster attractor that stems from the foliation of a period-three window of the logistic map [8].

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0375-9601/01/$ – see front matter
PII: S0375-9601(01)00404-2

Hence, it is sometimes necessary to use forms of coupling which could include nonlocal interactions, as for extended systems where the interaction strength decays slowly with the distance along the lattice. Lattices of maps with nonlocal coupling may be regarded as resulting from discretization of some partial integro-differential equations modeling physicochemical reactions [9]. This is the case, for example, in models of assemblies of biological cells with oscillatory activity, when their interaction is mediated by some rapidly diffusing chemical substance [10]. In systems of diffusion coupling in nucleation kinetics, where one eliminates the rapidly diffusing components, nonlocal couplings may also appear [11]. Two such prescription have attracted much attention: intermediate range couplings, that considers a finite number of non-nearest neighbors, usually with the same weight [12], and infinite range ones, as a power-law coupling, in which the coupling strength decays with the lattice distance r as a power-law r^{α}. This type of coupling has been proposed in models of some biological neural networks [13]. Small-world networks of coupled maps have been also intensively studied in recent years since the seminal work of Watts and Strogatz [14]. In such networks we have some regular couplings with nearest and non-nearest neighbors as well as a small number of randomly chosen nonlocal interactions [15,16].

In this Letter we will consider a one-dimensional lattice of maps with a power-law coupling, in such a way that we pass continuously from a nearest-neighbor to a mean-field coupling by varying a range parameter [17,18]. It turns out that some dynamical properties of CML, like frequency synchronization, are strongly dependent on the coupling range. We have found, for a lattice of circle maps with randomly distributed natural frequencies, that there is a phase transition between a synchronized and a nonsynchronized state as we go from a global to a local coupling [18,19].

If each of the maps presents chaotic dynamics when uncoupled, we naturally ask what are the possible effects of coupling on the individual behavior. Such a characterization is given by the Lyapunov spectrum of exponents for the system as a whole. The Lyapunov spectrum of a lattice of piecewise linear maps $x \mapsto f(x) = \beta x \pmod 1$ and logistic maps with local coupling has been studied by Kaneko [20], and by Isola et al. [21], that also studied the relationship between the Lyapunov spectrum for this kind of coupled map lattice and the spectrum of the discrete Schrödinger operator in quantum mechanics, a fact that was already pointed out by Ruelle [22]. Although there are some theoretical results on the qualitative shape of these spectra [23], there is no complete theory for this subject, even in simple cases.

In this Letter we present numerical results about the dependence of some Lyapunov spectrum properties on the coupling parameters for a lattice of coupled chaotic logistic maps, namely the maximal Lyapunov exponent and the average of the positive exponents. The coupling parameters to be considered are the strength and range. We point out that our results are in accordance with previous studies done for the short and long range cases.

Let us consider a lattice of N coupled logistic maps $x \mapsto f(x) = 1 - rx^2$, where $x_n^{(i)} \in [-1, +1]$ represents the state variable for the site i $(i = 1, 2, \ldots, N)$, at time n, and $r \in [0, 2]$. A power-law coupling is given by [19]

$$x_{n+1}^{(i)} = (1-\epsilon) f\left(x_n^{(i)}\right) + \frac{\epsilon}{\eta(\alpha)} \sum_{j=1}^{N'} \frac{1}{j^{\alpha}} \left[f\left(x_n^{(i+j)}\right) + f\left(x_n^{(i-j)}\right)\right], \quad (1)$$

where $\epsilon > 0$ and $\alpha > 0$ are the coupling strength and range, respectively, and $\eta(\alpha) = 2\sum_{j=1}^{N'} j^{-\alpha}$ is a normalization factor, with $N' = (N-1)/2$. This coupling term is actually a weighted average of discretized spatial second derivatives, the normalization factor being the sum of the corresponding statistical weights.

If $\alpha \to \infty$ only those terms with $j = 1$ will contribute to the summations present in the coupling term, and $\eta \to 2$, so that we get the usual future Laplacian coupling which connects nearest-neighbors only [1,2]. In the case where $\alpha = 0$, we have that $\eta = N - 1$ and the coupling becomes of a global type, connecting every site with the mean value of all lattice sites, regardless of their relative positions ("mean-field" model) [6]. The coupling in Eq. (1) may be regarded as a kind of interpolating form between these limiting cases. In the following we adopt $r = 2$ to ensure that, at least when uncoupled ($\epsilon = 0$), the maps are behaving chaotically, with Lyapunov exponent $\lambda_U = \ln 2$.

A lattice with N coupled one-dimensional maps is a N-dimensional discrete dynamical system: $x_{n+1}^{(i)} = \mathcal{F}^{(i)}(x_n^{(1)}, \ldots, x_n^{(N)})$ $(i = 1, \ldots, N)$. The corresponding Lyapunov spectrum is formed by N exponents, one for each independent eigendirection $\mathbf{u}_i$ in the tangent space: $\lambda_1 = \lambda_{\max} > \lambda_2 > \cdots > \lambda_N$. The Lyapunov exponent corresponding to the eigendirection $\mathbf{u}_i$ is given by

$$\lambda_i = \lim_{n\to\infty} \left(\frac{1}{n}\right) \ln \|\mathbf{A}_n \cdot \mathbf{u}_i\|, \tag{2}$$

where $\mathbf{A}_n = \prod_{\ell=1}^{n} \mathbf{J}_\ell$, with $[\mathbf{J}_\ell]_{ij} = \partial\mathcal{F}^{(i)}/\partial x_\ell^{(j)}$. The Jacobian matrix elements are given, for Eq. (1), by $[\mathbf{J}_\ell]_{ij} = \mathcal{H}_{ij}(\epsilon,\alpha) f'(x_\ell^{(j)})$, where

$$\mathcal{H}_{ij}(\epsilon,\alpha) = \begin{cases} 1-\epsilon, & \text{if } i = j, \\ \epsilon|i-j|^{-\alpha}/\eta(\alpha), & \\ \quad \text{if } i \neq j \text{ and } |i-j| \leqslant N', & \\ \epsilon(N-|i-j|)^{-\alpha}/\eta(\alpha), & \\ \quad \text{if } i \neq j \text{ and } |i-j| > N', & \end{cases} \tag{3}$$

and the primes denote differentiation with respect to the argument.

The maximal Lyapunov exponent $\lambda_{\max}$ represents the exponential rate at which an arbitrarily small displacement is amplified, so it suffices that $\lambda_{\max} > 0$ to, technically speaking, ensure that the lattice dynamics is chaotic. The sum of all exponents $\lambda_1 + \lambda_2 + \cdots + \lambda_N$ yields the rate of growth of a N-directional Euclidean volume element. In a system of coupled chaotic maps, it may well happen that many exponents are positive, hence a quantity of interest is the lattice-averaged value of the positive Lyapunov exponents [24]:

$$h = \langle\lambda_i\rangle_{i,\lambda_i>0} = \frac{1}{N}\sum_{i=1}^{\lambda_i>0} \lambda_i. \tag{4}$$

Pesin has proved that, if $f: M \to M$ is a diffeomorphism with an invariant ergodic measure absolutely continuous with respect to the Lebesgue measure of M, then this quantity is equal to the density of Kolmogorov–Sinai (KS) entropy, which is the asymptotic rate of creation of information by successive iterations of f [25]. It turns out that h is an upper bound for the KS entropy if f is a C^1 map preserving an ergodic measure [26]. The equality between h and the density of KS entropy is generally valid for systems — like Axiom A systems — having a Sinai–Ruelle–Bowen (SRB) measure, that is, a measure smooth along unstable directions [22].

We analyze the dependence of the average positive Lyapunov exponent and the maximal Lyapunov exponent on the parameters characterizing a power-law coupling: its strength ϵ and range α. Different profiles of randomly chosen initial conditions would give slightly different results for h, so we compute the mean value of h from a given number of random initial patterns. The value of $\langle h\rangle$ increases with α for a fixed value of ϵ (Fig. 1(a)). Although some fluctua-

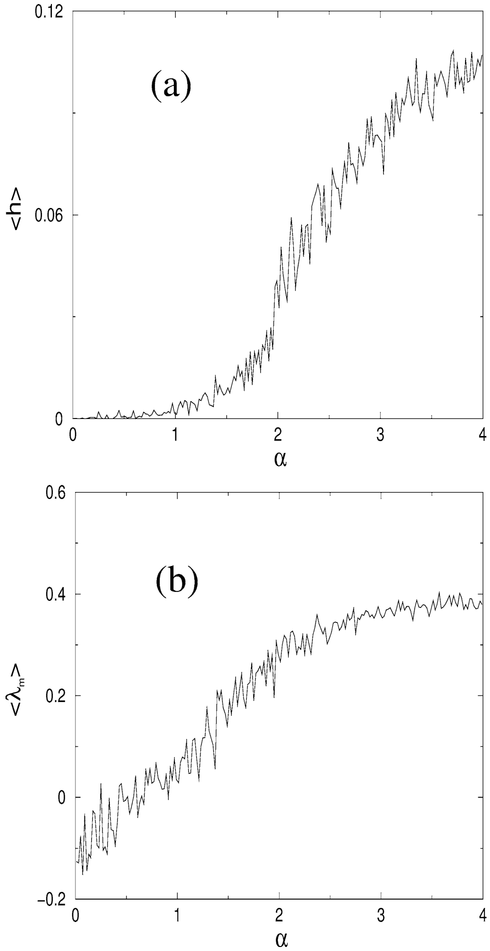

Fig. 1. (a) Average positive Lyapunov exponent and (b) maximal Lyapunov exponent versus coupling range α for a lattice of $N = 49$ coupled logistic maps with $r = 2.0$ and $\epsilon = 0.3$. We take the mean value of five random initial conditions patterns, with periodic boundary conditions.

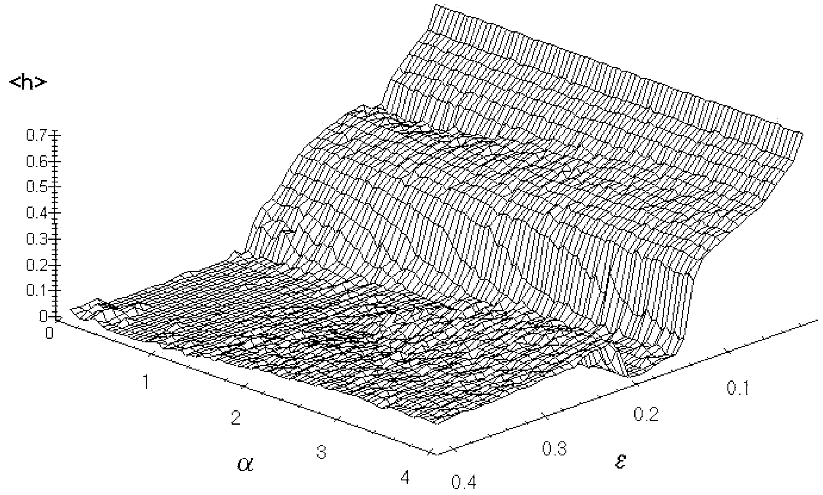

Fig. 2. Average positive Lyapunov exponent versus coupling parameters: strength ϵ and range α, for a lattice of $N = 49$ coupled logistic maps with $r = 2.0$. We take the mean value of five random initial conditions patterns, with periodic boundary conditions.

tions are still present, thanks to the finite size of the lattice and some statistical uncertainty, the general trend is a monotonic increase as the coupling range varies from global to local. A similar behavior is exhibited by the mean value of the maximal Lyapunov exponent $\langle\lambda_{max}\rangle$, that saturates for α higher than 3.0, because the large-alpha limit in the coupling is already reached even with such modest values (Fig. 1(b)).

The cases shown in Figs. 1(a) and (b) are representative of a moderate coupling strength. The dependence of the average positive exponent with ϵ and α is depicted in Fig. 2. For $\epsilon = 0$ (uncoupled maps) we have that $h = \lambda_U \approx 0.69$. We have a rough transition line at weak coupling ($0.15 \lesssim \epsilon \lesssim 0.2$) between very small values of h, indicating weak or no chaos, and a smooth increase characteristic of turbulent states of spatio-temporal chaos. This transition line has a mild inclination toward small ϵ-values as we increase the coupling range, indicating that the weakly chaotic regime is more prevalent for local couplings. The existence of this transition line can be explained with the help of a specific calculation of the Lyapunov spectrum that is feasible in the case $\alpha = 0$, and a so-called coherent attractor, for which $x_n^{(i)} = x_n^{(j)}$ for all i and j. In this case the Lyapunov spectrum is given by $\lambda_1 = \ln\gamma$ and $\lambda_j = \ln[\gamma(1-\epsilon)]$, for $j > 1$, with $\gamma = \exp(\lambda_U)$, and λ_U is the Lyapunov exponent of the uncoupled maps in the coherent attractor [6].

In our case $\lambda_U = \ln 2$, so that the $(N-1)$-fold degenerate Lyapunov spectrum is given by $\lambda_1 = \ln 2$ and $\lambda_j = \ln[2(1-\epsilon)]$, with $j = 2, 3, \ldots, N$. For $\epsilon > 1/2$ it turns out that only λ_1 is positive, so the average positive exponent is typically very small, vanishing for $N \to \infty$. On the other hand, for $0 < \epsilon < 1/2$ all exponents are positive, and we have

$$h = \frac{1}{N}\left\{\ln 2 + (N-1)\ln\bigl(2(1-\epsilon)\bigr)\right\}. \qquad (5)$$

For large N, it follows that h depends on ϵ as $\ln[2(1-\epsilon)]$ for $\epsilon < 1/2$, and vanishes for $\epsilon > 1/2$, which defines a critical point at $\epsilon = 1/2$ (Fig. 3). An inspection of Fig. 2 shows that, for $\alpha = 0$, there is a critical point (which is the intersection between the transition line and the line $\alpha = 0$) at $\epsilon \approx 0.2$, and the dependence with ϵ resembles of that estimated here. Hence the globally coupled case, while not being a coherent attractor, presents qualitatively the same critical behavior for the average positive Lyapunov exponent.

At the same time, another front of increase in h crops up in Fig. 2 at $\alpha \approx 1.0$, and there is a second rough transition line indicating a steeper increase of h, but with final values not very large, in comparison with those values achieved by the other front. The transition line has also a larger inclination as α builds up. The fact that this second transition line starts at

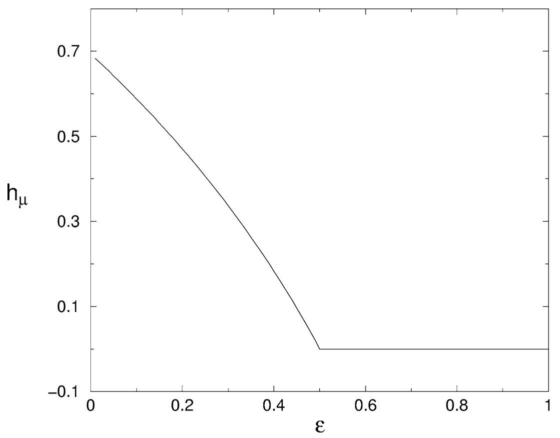

Fig. 3. Average positive Lyapunov exponent as a function of ϵ for a coherent attractor in a lattice of $N = 1001$ coupled logistic maps with $r = 2.0$ and global (mean-field) coupling.

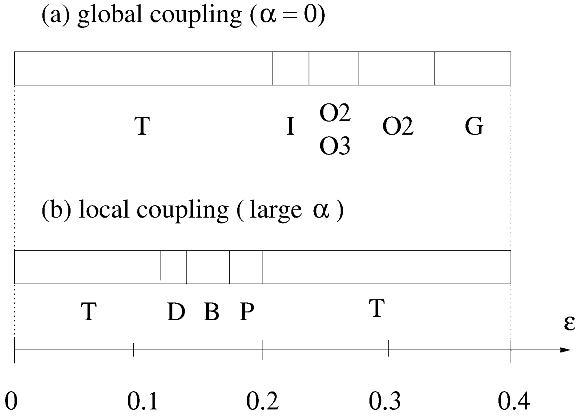

Fig. 4. Spatio-temporal patterns for a coupled logistic lattice with $r = 2.0$ for some values of coupling strength in the cases of (a) global mean-field [3] and (b) local nearest-neighbor couplings [6]. Patterns are identified by the following symbols: (T) turbulent; (I) space–time intermittency; (O2), (O3) ordered phase with 2 and 3 dominant clusters, respectively; (G) glassy; (D) defect turbulence; (B) Brownian motion of defects; (P) pattern selection.

$\alpha \approx 1.0$ is suggestive, since for this value of the coupling range a phase transition occurs between synchronized and nonsynchronized states [18,19]. Although this evidence — for lattices of sine-circle maps — was found for frequency synchronization, we observed that a similar synchronization occurs for the state variables themselves. A nonsynchronized and chaotic state would be related to nonzero values of h. However, since nonsynchronization does not imply necessarily a complete absence of spatial correlation, the average positive Lyapunov exponents do not have to be as high as in turbulent states.

A key to understand some features of Fig. 2 is the phase diagrams that are available in the literature for the global and local cases [3,6]. We present in Figs. 4(a) and (b) cross sections (at $r = 2.0$) of the phase diagrams for the global mean-field ($\alpha = 0$) and the local (large α) coupling cases, respectively. The global case presents a large turbulent phase (named T in Fig. 4(a)) for $0 < \epsilon < 0.207$, corresponding in Fig. 2 to the large region of high h-values. In the interval $0.207 < \epsilon < 0.238$ we have spatio-temporal intermittency (I). Further values of the coupling strength exhibit ordered phases with two and three dominant clusters (O2 and O3) when $0.238 < \epsilon < 0.339$, and finally a glassy phase (G) for $\epsilon > 0.339$. All of them but the last are characterized by small average positive exponents. The slight increase of h is compatible with the existence of the glassy phase.

In the case of local coupling we may take $\alpha = 4$ in Fig. 2 for comparison. It is unlikely that some novel feature would occur for higher coupling ranges, since for α larger than ≈ 3.0 the coupling is essentially between nearest neighbors only. For $0 < \epsilon < 0.117$ we have a fully developed turbulent state (T) in the phase diagram of Fig. 4(b). This corresponds to the region characterized by higher h-values, just like in the global coupling case. The interval $0.117 < \epsilon < 0.200$, for which there is a kind of valley of small or vanishing h in Fig. 2, corresponds to defect (weak chaos) turbulence (D) for $0.117 < \epsilon < 0.138$, Brownian motion of defects (B) for $0.138 < \epsilon < 0.172$ and pattern selection with selected domains of sizes one and two (P) for $0.172 < \epsilon < 0.200$. We have numerically verified that this valley continues to exist for α as high as 400. This is followed, for higher coupling strength ($\epsilon > 0.200$), by a new turbulent region (T), that is just the second front of nonvanishing h. The smaller values observed in this second front are due to the stronger coupling that prevents the existence of a large number of chaotic attractors.

Up to the second transition line that starts in $\alpha \approx 1.0$, and which is probably due to a loss of synchronization, we pass rather smoothly from the phase diagrams of a locally to a globally coupled lattice.

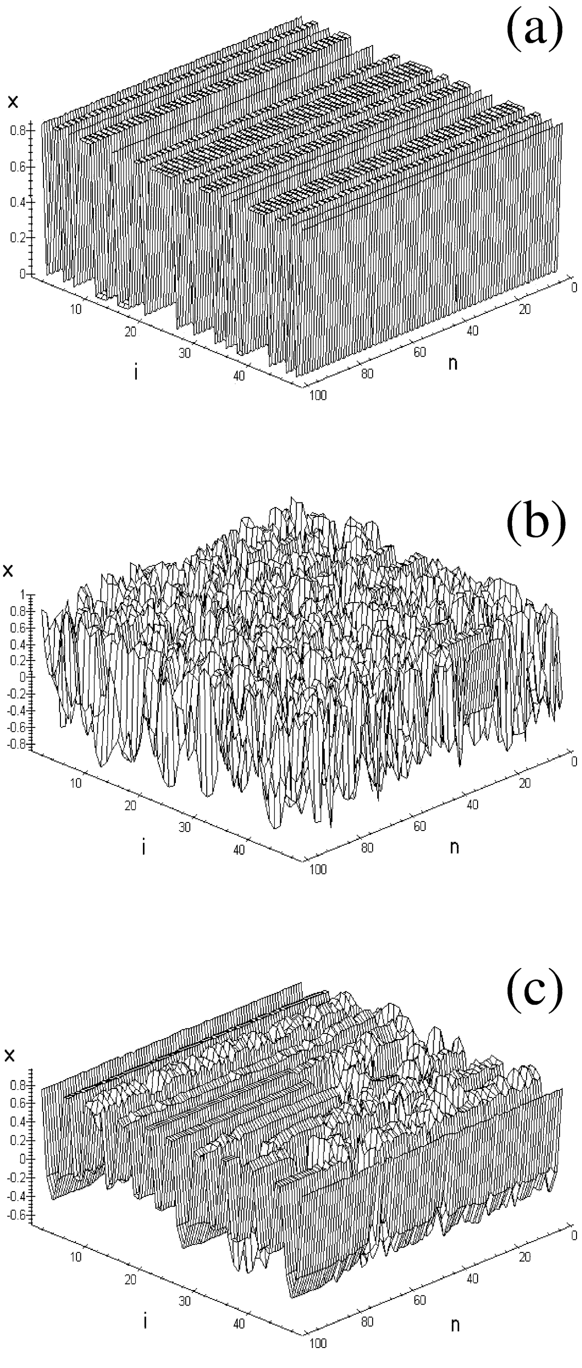

Fig. 5. Space–time–amplitude plots for a lattice of $N = 49$ coupled logistic maps with $r = 2.0$, $\epsilon = 0.3$, random initial conditions and periodic boundary conditions, for coupling ranges: (a) $\alpha = 0.0$, (b) $\alpha = 3.0$, and (c) $\alpha = 2.0$.

In Fig. 5 we depict some space–time–amplitude diagrams that illustrate some of the features we just mentioned, in the case of moderate coupling ($\epsilon = 0.3$). In Fig. 5(a) we have a mean-field ($\alpha = 0.0$) lattice. From Fig. 4(a) we know that it represents an ordered phase with dominant clusters. Fig. 5(b) represents a locally coupled ($\alpha = 3.0$) lattice. The corresponding phase diagram suggest that it represents a fully developed turbulent state, which is seen in the apparent lack of space and time correlations. In Fig. 5(c) we depict a situation in which there is intermittency in both space and time, for an intermediate coupling ($\alpha = 2.0$). In the diagram of Fig. 2 we can locate this case in the second front of positive h-values, which is more typical of a local coupling. However, for large α, the scheme of Fig. 4(b) does not include space–time intermittency, hence this observed feature may either disappear with increasing α or survive in a very tiny region of the parameter space.

To summarize our results, the average positive Lyapunov exponent, that is a measure of randomness in a lattice of coupled chaotic maps, as well as the maximal Lyapunov exponent, depend on the coupling properties in a nontrivial manner. We have introduced a range parameter that indicates the decay rate of the interactions between non-nearest neighbor lattice sites. Roughly speaking, the degree of randomness decreases with the coupling strength for any positive range. However, for global (low-α) lattices this decrease is less pronounced than for local (high-α) lattices. In the latter case there is a second front of moderate randomness that arises from $\alpha \approx 1$. In the extreme cases of $\alpha = 0$ and large α our results agree with the phase diagrams already drawn for logistic lattices. The spatio-temporal behavior between these limiting cases is partially explained by the synchronization properties of the lattice.

Acknowledgements

This work was made possible by partial financial support from CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and Fundação Araucária (State of Paraná, Brazil). R.L.V. wishes to thank Professor Celso Grebogi (University of São Paulo, Brazil) for valuable discussions and suggestions.

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