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LETTER TO THE EDITOR

Comment on a Hamiltonian representation for helically symmetric fields

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Abstract. A general scheme for Hamiltonian representation of magnetic field flow is applied to the case of helical symmetry .

Some years ago, Turner (1985) presented a Hamiltonian formulation for magnetic field-line flow in the case when the magnetic field is helically symmetric, i.e. it depends on the variables r and $\chi = m\theta + kz$, where (r, θ, z) are the usual cylindrical coordinates. Such fields occur very frequently in plasma physics applications, in both fusion and astrophysical plasmas. I would like to show a different approach to this subject, since Turner's derivation is also obtainable from a general formulation which deals with general systems of coordinates (Whiteman 1977). This method has been applied to Mercier coordinates (in expansions near the magnetic axis) (Bernardin and Tataronis 1985), and more recently, to cylindrical equilibria (Viana 1993).

Let us consider a magnetostatic equilibrium described by the set of contravariant coordinates (x^1, x^2, x^3), where the magnetic field $\boldsymbol{B} = (B^1, B^2, B^3)$ exibits symmetry with respect to the x^3 coordinate. The magnetic field-line equations can be cast in a Hamiltonian form after definition of the following canonically conjugated variables:

$$q = x^1 \tag{1}$$

$$p = \int \mathrm{d}x^2 \sqrt{g} B^3(x^1, x^2) + \gamma(x^1, x^3) \tag{2}$$

where $g = \det g_{ij}$ (g_{ij} being the covariant metric tensor), $B^i (i = 1, 2, 3)$ are the contravariant field components, and γ is an arbitrary function.

Due to the spatial symmetry, x^3 plays the role of a time variable, so that our Hamiltonian system of equations ($\mathrm{d}q/\mathrm{d}t = \partial\mathcal{H}/\partial p$ and $\mathrm{d}p/\mathrm{d}t = -\partial\mathcal{H}/\partial q$) is not actually dynamical, since the field configuration is magnetostatic. The field-line Hamiltonian reads

$$\mathcal{H} = \int \mathrm{d}x^2 \sqrt{g} B^1(x^1, x^2) + \delta(x^1, x^3) \tag{3}$$

and the functions in (2) and (3) must satisfy the following constraint relation:

$$\sqrt{g} B^2(x^1, x^2) + \frac{\partial\mathcal{H}(x^1, x^2, x^3)}{\partial x^1} + \frac{\partial p(x^1, x^2, x^3)}{\partial x^3} = 0 \, . \tag{4}$$

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In order to apply these formulae to helically symmetric fields, we choose $x^1 = r$, $x^2 = \chi$ and $x^3 = z$. The covariant metric tensor is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2/m^2 & -kr^2/m^2 \\ 0 & -kr^2/m^2 & 1 + k^2r^2/m^2 \end{pmatrix} \tag{5}$$

so that $g = r^2/m^2$. Whereas the canonical coordinate is $q = r$, its conjugated momentum, according (2), is

$$p(r, \chi, z) = \frac{1}{m} \int_0^{\chi} \mathrm{d}\chi' r B_z(r, \chi') + \gamma(r, z) \,. \tag{6}$$

If $\gamma \equiv 0$ the above result is the same as that obtained by Turner, up to the (unessential) factor m^{-1}. Equation (3) gives the field-line Hamiltonian, whose χ-derivative is equal to

$$\frac{\partial \mathcal{H}}{\partial \chi} = \frac{r}{m} B_r(r, \chi) \tag{7}$$

and the constraint relation (4) furnishes

$$\frac{\partial \mathcal{H}}{\partial r} = -\frac{1}{m} \left(m B_\theta(r, \chi) + kr B_z(r, \chi) \right) \tag{8}$$

provided we set $\gamma = \gamma(r)$. These equations, up to the same factor, are in accordance with the results of Turner (1985), which have been obtained through a different approach.

References

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