Gas flow around a longitudinally oscillating plate at arbitrary ratio of collision frequency to oscillation frequency

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Abstract. A gas flow around a longitudinally oscillating plate is considered on the basis of the kinetic equation. The main parameter determining the problem solution is the ratio of intermolecular collision frequency to the oscillation frequency. Two methods of computation were used: the integro-moment method and the discrete velocity one. Diffuse gas-surface interaction law was assumed. The numerical calculations were carried out for a wide range of the ratio of collision frequency to oscillation frequency.

INTRODUCTION

Stationary solutions of the Boltzmann kinetic equation and its models were studied profoundly during several last decades, see e.g. [1, 2, 3], but it was paid less attention to time periodic solutions. Some analytical solutions for one-dimensional time-dependent flows assuming the diffuse gas-surface interaction law were obtained by Cercignani [1, 4]. An oscillatory Couette flow was studied numerically in Refs.[5, 6] by the direct simulation Monte Carlo method. However, this method does not provide reliable results for such kind of flows because of the significant statistical scattering. In the present paper we study a behavior of a gas flow near a flat plate oscillating in its own plane on the basis of the kinetic equation. Such a flow is realized in micro-accelerometers, inertial sensors, resonant filters, etc.

The problem can be easily solved on the basis of the Navier-Stokes equation, see e.g. Ref.[7], which is valid under two conditions. First, the molecular mean free path ℓ must be significantly smaller than a characteristic size a of gas flow. Second, the molecular mean-free-time must be significantly smaller than a characteristic time of gas flow. In case of oscillating flows the second condition can be expressed in terms of the intermolecular collision frequency ν and oscillation frequency ω. To characterize the speed of oscillation we will use the following criterion

$$\theta = \frac{\nu}{\omega}. \tag{1}$$

If $\theta \to \infty$ many intermolecular collisions occur during one cycle of the oscillation. This regime will be called as low oscillation speed. Under such a condition the Navier-Stokes equation is applicable. Thus, this regime can be also called as hydrodynamic. In the other limit, i.e. $\theta \to 0$, very few intermolecular collisions occur during one oscillation cycle. This regime intermolecular collisions can be neglected and the problem can be solved analytically on the level of the velocity distribution function. The regime $\theta \sim 1$ is called transitional with respect to the oscillation speed. Under this condition we can neither apply the hydrodynamic equations nor neglect by the intermolecular collision, but the problem must be solved on the basis of the non-stationary kinetic equation.

To make the use of the criterion (1) more convenient the intermolecular collision frequency should be expressed in terms of measurable quantities. As is known the frequency ν has the order of P/μ, where P is a pressure of the gas and μ is its viscosity. Thus, the oscillation speed parameter can be defined as

$$\theta = \frac{P}{\mu\omega}. \tag{2}$$

The aim of the present paper is to calculate the gas flow near an oscillating plate over a wide range of the oscillation speed parameter θ.

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STATEMENT OF THE PROBLEM

Consider a monoatomic gas occupying a semi-infinite space $x' > 0$. An infinite flat plate fixed at $x' = 0$ and parallel to the yz-plane oscillates harmonically in the y direction with frequency ω, i.e. the velocity of the plate depends on the time t' as

$$U_{\text{w}} = \Re[U_m \exp(-i\omega t')], \tag{3}$$

where the symbol $\Re$ denotes the real part of complex expression. The quantity U_m is the velocity amplitude, which is assumed to be small when compared with the most probable molecular velocity v_m of the gas, i.e.

$$U_m \ll v_m, \qquad v_m = \left(\frac{2kT}{m}\right)^{1/2}, \tag{4}$$

where m is the molecular mass, T is the gas temperature and k is the Boltzmann constant.

The plate oscillation causes a flow of the gas, namely, the bulk velocity U_y of the gas in the y-direction and the shear stress P_{xy} depend on the time harmonically

$$U_y(t',x') = \Re\left[\tilde{U}_y(x')\exp(-i\omega t')\right], \qquad P_{xy}(t',x') = \Re\left[\tilde{P}_{xy}(x')\exp(-i\omega t')\right], \tag{5}$$

where $\tilde{U}_y(x')$ and $\tilde{P}_{xy}(x')$ are complex quantities.

For convenience, the dimensionless time, coordinate, bulk velocity and shear stress are introduced as

$$t = \omega t', \qquad x = \frac{\omega}{v_m}x', \qquad u(x) = \frac{\tilde{U}_y}{U_m}, \qquad \Pi(x) = \frac{\tilde{P}_{xy}}{2P}\frac{v_m}{U_m}, \tag{6}$$

respectively. Note, the quantities $u(x)$ and $\Pi(x)$ are complex and can be written down as

$$u(x) = u_m(x)\exp[i\varphi_{\text{u}}(x)], \qquad \Pi(x) = \Pi_m(x)\exp[i\varphi_{\text{p}}(x)], \tag{7}$$

where $u_m(x)$ and $\Pi_m(x)$ are the amplitude of the bulk velocity and shear stress, respectively, and $\varphi_{\text{u}}(x)$ and $\varphi_{\text{p}}(x)$ are their phases. In practice, it is interesting to know the attenuations defined as

$$a_u = -\frac{\mathrm{d}\ln u_m}{\mathrm{d}x}, \quad a_{\text{p}} = -\frac{\mathrm{d}\ln \Pi_m}{\mathrm{d}x}. \tag{8}$$

The quantities defined above will be obtained in a wide range of the oscillation speed parameter θ.

LOW OSCILLATION SPEED REGIME

At the slow speed of oscillation (hydrodynamic regime) the Navier-Stokes equation [8] is employed, which in our notations reads

$$2i\theta u + \frac{\partial^2 u}{\partial x^2} = 0. \tag{9}$$

If we assume the non-slip boundary conditions, i.e.

$$u = 1 \quad \text{at} \quad x = 0, \tag{10}$$

then Eq.(9) has the following solution [8]

$$u(x) = \exp[(i-1)\sqrt{\theta}x]. \tag{11}$$

The shear stress Π of the gas flow is obtained by using the Newton law

$$\Pi(x) = -\frac{1}{2\theta}\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\theta}}\exp\left[(i-1)\sqrt{\theta}x - i\frac{\pi}{4}\right]. \tag{12}$$

From (11) and (12) the attenuations and phases are easily obtained

$$a_u(x) = a_{\text{p}}(x) = \sqrt{\theta}, \qquad \varphi_{\text{u}}(x) = \sqrt{\theta}x, \qquad \varphi_{\text{p}}(x) = \sqrt{\theta}x - \frac{\pi}{4}. \tag{13}$$

Thus, the attenuations are constant and the phases linearly depend on the coordinate x.

KINETIC EQUATION

For an arbitrary value of the oscillation speed parameter θ the problem must be solved on the level of the velocity distribution function $f(t', \mathbf{r}', \mathbf{v})$, which obeys the Boltzmann kinetic equation. Here, $\mathbf{r}'$ is the position vector, and $\mathbf{v}$ is the molecular velocity. For the problem in question the distribution function does not depend on the coordinates y' and z'. Thus, the Boltzmann equation reads

$$\frac{\partial f}{\partial t'} + v_x \frac{\partial f}{\partial x'} = Q(f f_*), \tag{14}$$

where $Q(f, f_*)$ is the collision integral [1, 2, 3, 9, 10]. Till now, numerical calculations based on the exact Boltzmann equation require significant computational efforts, while the model equations provide reliable results with modest efforts. As was shown in Ref.[3] the Bhatnagar, Gross, Krook (BGK) [11] model equation is appropriate for isothermal gas flows. Thus, for the problem under consideration we employ this model, which reads

$$Q(f f_*) = \frac{P}{\mu}\left(f^M - f\right), \tag{15}$$

where f^M is the local Maxwellian function

$$f^M = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m(\mathbf{v}-\mathbf{U})^2}{2kT}\right], \tag{16}$$

n, $\mathbf{U}$, and T are local number density, bulk velocity, and temperature of the gas, respectively. Under the assumption (4) we may consider that the density n, pressure P, and temperature T are constant, while the bulk velocity $\mathbf{U}$ has only the y component expressed via the distribution function as

$$U_y(t', x') = \frac{1}{n}\int v_y f(t', x', \mathbf{v})\, \mathrm{d}\mathbf{v}. \tag{17}$$

Moreover, we are interested in the shear stress of the gas calculated as

$$P_{xy}(t', x') = m\int v_x v_y f(t', x', \mathbf{v})\, \mathrm{d}\mathbf{v}. \tag{18}$$

Because of the condition (4) the Boltzmann equation (14) can be linearized by representing the distribution function as

$$f(t', x', \mathbf{v}) = \frac{n}{(\sqrt{\pi} v_m)^3} \exp(-c^2)\left[1 + h(t, x, \mathbf{c})\frac{U_m}{v_m}\right], \tag{19}$$

where $h(t, x, \mathbf{c})$ is the perturbation function and $\mathbf{c} = \mathbf{v}/v_m$ is the dimensionless molecular velocity.

Substituting Eq.(19) into (14) with (15) we obtain the linearized BGK equation in the dimensionless form

$$\frac{\partial h}{\partial t} + c_x \frac{\partial h}{\partial x} = \theta\left(2c_y \Re u - h\right). \tag{20}$$

The perturbation function can be represented as

$$h(t, x, \mathbf{c}) = \Re\left[\tilde{h}(x, c_x, c_y)\mathrm{e}^{-it}\right], \tag{21}$$

Then Eq.(20) is written down in terms of the complex functions $\tilde{h}$ and u as

$$(\theta - i)\tilde{h} + c_x \frac{\partial \tilde{h}}{\partial x} = 2\theta c_y u. \tag{22}$$

With the help of Eqs.(6), (17) - (19) and (21) we obtain.

$$u(x) = \frac{1}{\pi^{3/2}}\int \exp(-c^2)\tilde{h}(x, c_x, c_y) c_y\, \mathrm{d}\mathbf{c}, \qquad \Pi(x) = \frac{1}{\pi^{3/2}}\int \exp(-c^2)\tilde{h}(x, c_x, c_y) c_x c_y\, \mathrm{d}\mathbf{c}. \tag{23}$$

We assumed the diffuse scattering boundary condition on the oscillating surface, i.e.

$$h = \Re[2c_y e^{-it}] \qquad \text{at} \qquad x = 0. \tag{24}$$

To eliminate the variable c_y a new perturbation function is introduced as

$$\Phi(x,c_x)=\frac{1}{\pi^{1/2}}\int c_y\tilde{h}(x,c_x,c_y)\exp\left(-c_y^2\right)\mathrm{d}c_y \tag{25}$$

Multiplying Eq.(22) by $c_y\exp\left(-c_y^2\right)/\pi^{1/2}$ and integrating it with respect to c_y we obtain

$$(\theta-i)\Phi+c_x\frac{\partial\Phi}{\partial x}=\theta u. \tag{26}$$

In terms of the function Φ the boundary condition (24) reads

$$\Phi=1 \qquad \text{at} \qquad x=0 \qquad \text{and} \qquad c_x>0. \tag{27}$$

Eq. (26) with the boundary condition (27) was solved by two different methods of computation: integro-moment and discrete velocity. The first method allow us to obtain the following integral equations for the moments of the distribution function

$$u(x)=\frac{1}{\sqrt{\pi}}I_0[(\theta-i)x]+\frac{\theta}{\sqrt{\pi}}\int_0^\infty u(\xi)I_{-1}[(\theta-i)|\xi-x|],\mathrm{d}\xi. \tag{28}$$

$$\Pi(x)=\frac{1}{\sqrt{\pi}}I_1[(\theta-i)x]+\frac{\theta}{\sqrt{\pi}}\int_0^\infty u(\xi)\mathrm{sign}(\xi-x)I_0[(\theta-i)|\xi-x|]\,\mathrm{d}\xi, \tag{29}$$

where the functions $I_n(x)$ are defined as

$$I_n(z)=\int_0^\infty c^n\exp\left(-c^2-\frac{z}{c}\right)\mathrm{d}c. \tag{30}$$

Eqs. (28) and (29) were solved by a direct numerical method described in Sec.2.12.4 of Ref.[3].

In the discrete velocity method we used a regularly distribution of points in physical space and a non-regularly distribution of points in velocity space (higher density for small values of c_x and lower density for its large values). The kinetic equation (26) is approximated by a finite difference scheme and the moments u and Π are calculated by using a quadrature formula. The numerical error is less than 0.1%.

HIGH OSCILLATION SPEED REGIME

At the high speed of oscillation ($\theta=0$) the solution can be obtained analytically. In this regime Eq.(22) is reduced to

$$-i\tilde{h}+c_x\frac{\partial\tilde{h}}{\partial x}=0. \tag{31}$$

Integrating this equation with respect x we obtain

$$\tilde{h}(x,c_x,c_y)=\begin{cases}2c_y\exp\left(i\frac{x}{c_x}\right), & c_x>0,\\ 0, & c_x<0,\end{cases} \tag{32}$$

where the boundary condition (24) was used. Substituting this solution into (23) the bulk velocity and shear stress are obtained as

$$u(x)=\frac{1}{\sqrt{\pi}}I_0(-ix), \qquad \Pi(x)=\frac{1}{\sqrt{\pi}}I_1(-ix). \tag{33}$$

It is impossible to obtain analytical expressions of the attenuations and phases based on Eq. (33). Thus, only limit values at $x=0$ and asymptotic behavior at $x\to\infty$ are given below. Using the power representation given by Eq.(27.5.4) of Ref.[12], we obtain the limit values of the attenuations and phases at the surface, i.e. at $x=0$

$$a_u=\sqrt{\pi}, \quad a_{\mathrm{p}}=0, \qquad \varphi_{\mathrm{u}}=\varphi_{\mathrm{p}}(x)=0. \tag{34}$$

At large distance ($x\gg1$) using Eq.(27.5.8) of Ref.[12] we have

$$a_u(x)=a_{\mathrm{p}}(x)=\frac{1}{(4x)^{1/3}}, \qquad \varphi_{\mathrm{u}}(x)=\frac{3\sqrt{3}}{2}\left(\frac{x}{2}\right)^{2/3}, \qquad \varphi_{\mathrm{p}}(x)=\varphi_{\mathrm{u}}(x)-\frac{\pi}{6}. \tag{35}$$

One can see that in this regime the solution is quite different from that for the low oscillation speed. First, the attenuations are not constant, but they slowly decrease by increasing the coordinate x. Second, the phases are not linear functions of the coordinate x, but they increase slower than the liner function of the coordinate x.

NUMERICAL RESULTS AND DISCUSSIONS

The results obtained by both methods are in agreement with each other within the numerical accuracy, i.e. 0.1%. The advantage of the integro-moment method is that only the x coordinate is discretized, while the molecular velocity is not. However, this method requires more computational memory and CPU time. The discrete velocity method requires modest computational efforts but the velocity space must be discretized.

The amplitudes u_m, Π_m and phases φ_u, φ_P at the surface, i.e. at $x = 0$, are given in Table 1. From these data we can see that the velocity amplitude u_m monotonely varies from 0.5 to 1 by varying the parameter θ from 0 to ∞. At the same time, the shear stress amplitude Π_m varies from the value $(2\sqrt{\pi})^{-1}$ and tends to zero as $(\sqrt{2\theta})^{-1}$ when θ tends to infinity. The velocity phase φ_u is equal to zero at both limits $\theta = 0$ and $\theta = \infty$. In the transitional regime $\theta \sim 1$ the phase φ_u has a maximum value. The phase of the shear stress varies from zero at the high oscillation speed ($\theta \ll 1$) to $-\pi/4$ at the low oscillation speed ($\theta \gg 1$).

TABLE 1. Amplitudes and phases at $x = 0$ for diffuse scattering

θ	u_m	Π_m	φ_u	$-\varphi_P$
0	0.5	0.2821	0.	0.
0.01	0.5000	0.2821	0.0025	0.0019
0.1	0.5008	0.2819	0.0250	0.0190
1.	0.5539	0.2688	0.1791	0.1662
5.	0.7291	0.2017	0.2062	0.4138
10.	0.7988	0.1625	0.1699	0.5063
20.	0.8531	0.1261	0.1319	0.5798
40.	0.8872	0.09490	0.1074	0.6321
60.	0.9078	0.08269	0.08235	0.6672
80.	0.9150	0.08014	0.08015	0.6823
∞	1.	$(2\theta)^{1/2}$	0.	$\pi/4$

The attenuations a_u and a_P characterize the behavior of the amplitude in the gas bulk. They are given in Fig. 1. This quantities vary sharply only in a small region near the surface. That is why it is presented only in the range $0 \le x \le 0.1$.

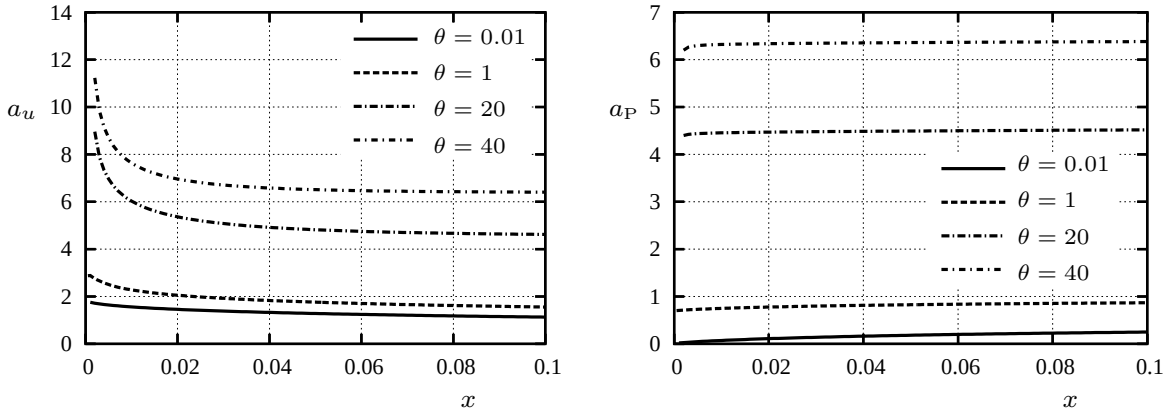

FIGURE 1. Attenuatins of the bulk velocity and shear stress vs distance x for the diffuse scattering

From Fig. 1 we conclude that: (i) The attenuation of both bulk velocity and shear stress increases by increasing the oscillation speed parameter. (ii) At the low oscillation speed $\theta = 80$ it tends to the constant value far from the surface. In accordance with Eq.(13) the constant value is $\sqrt{\theta}$. (iii) At the high oscillation speed $\theta = 0.01$ the attenuation is not constant but it slowly decreases by increasing the distance from the surface. (iv) The attenuation of the bulk velocity increase near the surface, while the attenuation of the shear stress slightly decreases.

The phases φ_u and φ_P are presented in Fig. 2 as function of the coordinate x.

It can be seen that: (i) The phases increase faster for higher value of the the oscillation speed parameter θ. (ii) At the low oscillation speed ($\theta = 80$) the phases φ_u and φ_P are linear functions of the coordinate x except a small region

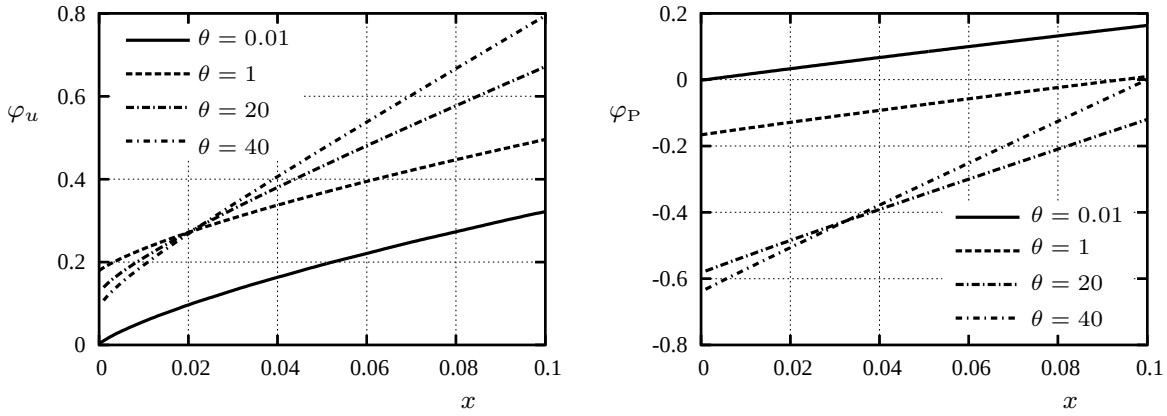

FIGURE 2. Phases of the bulk velocity and shear stress vs distance x for the diffuse scattering

near the surface. (iii) At the high oscillation speed $(\theta = 0.01)$ the phases increase slower than the linear functions, i.e. they increase as $x^{2/3}$.

CONCLUSION

Numerical calculations of the gas flow near a longitudinally oscillating plate were carried out in a wide range of the oscillation speed parameter θ, which is defined as a ratio of the intermolecular collision frequency to the oscillation one. The integro-moment and discrete velocity methods were applied to the BGK kinetic model of the Boltzmann equation. All quantities monotonely depend on the oscillation speed parameter, except the phase of the bulk velocity. The value of this quantity at the surface tends to zero in both limit $\theta = 0$ and $\theta = \infty$, but it reaches its maximum value in the transitional regime, i.e. at $\theta \sim 1$.

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