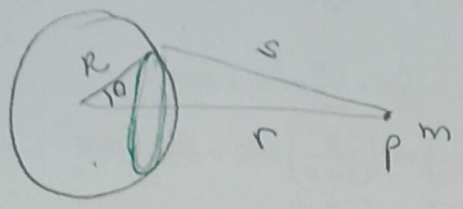
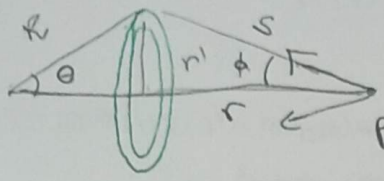
Teorema das cascas esféricas

Esfera oca, M, R, m, r

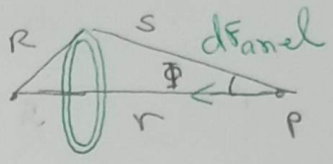




$r' = R \operatorname{sen}\theta$

$s^2 = R^2 + r^2 - 2Rr\cos\theta$

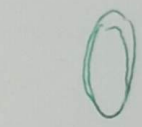
$$dF_{anel} = G\frac{dM\,m}{s^2}\cos\phi \quad (1)$$



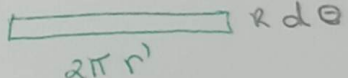
$$F = \int dF_{anel} \quad \text{(mesmo sentido)}$$

$$F = Gm\int \frac{dM}{s^2}\cos\phi \quad (2)$$

Relacionar dm, s e $\cos\Phi$:



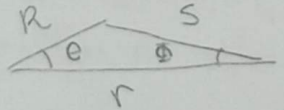
$$\frac{M_T}{A_T} = \frac{dM}{dA} \Rightarrow \frac{M}{4\pi R^2} = \frac{dM}{2\pi r' R\,d\theta}$$



$$\Rightarrow dM = \frac{M r' d\theta}{2R} \quad \text{mas } r' = R\operatorname{sen}\theta$$

$$\Rightarrow dM = \frac{M}{2}\operatorname{sen}\theta\, d\theta \Rightarrow F = \frac{GmM}{2}\int \frac{\operatorname{sen}\theta\cos\Phi}{s^2}d\theta \quad (3)$$

relacionando θ e Φ



$$R^2 = s^2 + r^2 - 2sr\cos\phi \quad (3.1)$$

$$s^2 = R^2 + r^2 - 2Rr\cos\theta \quad (3.2)$$

vamos derivar a relação ↗

$$2s\,ds = 0 + 0 - 2Rr(-\operatorname{sen}\theta)\,d\theta$$

$$\Rightarrow 2s\,ds = 2Rr\operatorname{sen}\theta\,d\theta$$

$$\Rightarrow \operatorname{sen}\theta\,d\theta = \frac{s}{rR}ds \quad (4)$$

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(4) ⇒ (3)

$$F = \frac{GmM}{2} \int \frac{\cos\Phi}{s^2} \frac{s}{rR}\, ds \quad \Rightarrow \quad F = \frac{GMm}{2rR} \int \frac{s\cos\Phi}{s^2}\, ds$$

De (3.1) temos $s\cos\Phi = \dfrac{s^2 + r^2 - R^2}{2r}$

Para $r > R$:

Logo:

$$F = \frac{GMm}{2rR(2r)} \int_{r-R}^{r+R} \left(1 + \frac{r^2 - R^2}{s^2}\right) ds$$

$$F = \frac{GMm}{4r^2R} \left[\int_{r-R}^{r+R} ds + (r^2 - R^2) \int_{r-R}^{r+R} s^{-2}\, ds \right]$$

$$= \frac{GMm}{4r^2R} \left[r + R - (r - R) + (r^2 - R^2) \frac{(-1)}{s} \Big|_{r-R}^{r+R} \right]$$

$$= \frac{GMm}{4r^2R} \left[2R - (r^2 - R^2) \left(\frac{1}{r+R} - \frac{1}{r-R} \right) \right]$$

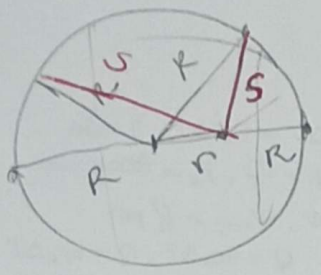
$$= \frac{GMm}{4r^2R} \left[2R - (r^2 - R^2) \left(\frac{r - R - r - R}{(r^2 - R^2)} \right) \right]$$

$$= \frac{GMm}{4r^2R} 4R \quad \Rightarrow \quad F = \frac{GMm}{r^2}$$

$$\Rightarrow F_{fora} = F_{pontual}$$

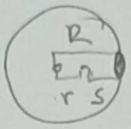
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Para $r < R$



$$F = \frac{GMm}{4r^2R}\left[\int_{R-r}^{R+r} ds + (r^2-R^2)\int_{R-r}^{R+r} s^{-2}\,ds\right]$$

menor $s \rightarrow R-r$



maior $s \rightarrow R+r$



$$F = \frac{GMm}{4r^2R}\left[R+r-(R-r) - (r^2-R^2)\left(\frac{1}{R+r} - \frac{1}{R-r}\right)\right]$$

$$F = \frac{GMm}{4r^2R}\left[2r - (r^2-R^2)\left(\frac{R-r-R-r}{R^2-r^2}\right)\right]$$

$(r^2-R^2) = -1$

$$F = \frac{GMm}{4r^2R}\left[2r-2r\right]$$

$$\Rightarrow F_{dentro} = 0 \quad \ddot{\smile}$$

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