Multistability and self-similarity in the parameter-space of a vibro-impact system

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Abstract
The dynamics of a dissipative vibro-impact system called impact-pair is investigated. This system is similar to Fermi-Ulam accelerator model and consists of an oscillating one-dimensional box containing a point mass moving freely between successive inelastic collisions with the rigid walls of the box. In our numerical simulations, we observed multistable regimes, for which the corresponding basins of attraction present a quite complicated structure with smooth boundary. In addition, we characterize the system in a two-dimensional parameter space by using the largest Lyapunov exponents, identifying self-similar periodic sets.

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1. INTRODUCTION

In order to study cosmic ray accelerated to high energy, Enrico Fermi proposed an accelerator model as a dynamical system [1], consisting of a classical particle interacting with a time dependent magnetic field. The original model was later modified and studied under different approaches. For example, the dynamics of the well-known Fermi-Ulam model have been investigated [2]. This model consists of a point mass moving between a rigid fixed wall and an oscillatory wall. In recent years, Fermi-Ulam model has attracted a significant attention [3–6].

In engineering context, systems similar to the dissipative Fermi-Ulam model have been intensively studied, like gearbox model [7–10] and impact damper [11, 12]. These systems, called vibro-impact or impact oscillator, appear in a wide range of practical problems, such as percussive drilling tools [13], print hammers [14], and vibro-impact moling systems [15], just to mention a few. For these systems a better understanding of their dynamics may help to reduce the negative effects of impacts and ultimately to improve practical designs.

In this work, we consider a prototypical vibro-impact system, known as impact-pair. This system is comprised of a ball moving between two oscillatory walls. Numerical studies have shown a rich dynamical behavior with several nonlinear phenomena observed, such as bifurcations, chaotic regimes [16, 17], crises [18], and basin hoppings [19]. In addition, control of chaotic dynamics can be applied to stabilize unstable periodic orbits embedded in the chaotic attractor [20]. We aim here to explore some dynamical properties with emphasis on characterization of basins of attraction with complicated smooth boundaries and identification of self-similar periodic sets called shrimps [21].

This paper is organized as follows. In Section 2 we present the model and the equations of motion for the impact-pair system. In Section 3, we investigate coexistence of different regimes and their basins of attraction. In Section 4, we characterize the impact-pair system in the two parameter space. The last section contains our conclusions.

2. MATHEMATICAL DESCRIPTION

In this section we present the basic equations of the impact-pair system [16, 17] shown schematically in Fig. 1. In addition, we describe how to obtain an impact map, also called
transcendental map. The map is useful to calculate the Lyapunov exponents [18].

The impact-pair system is composed of a point mass \( m \), whose displacement is denoted by \( x \), and an one-dimensional box with a gap of length \( \nu \). The mass \( m \) is free to move inside the gap and the motion of the box is described by a periodic function, \( \alpha \sin(\omega t) \).

In an absolute coordinate systems, equation of motion of the point mass \( m \) is given by:

\[
\ddot{x} = 0. \tag{1}
\]

Denoting the relative displacement of the mass \( m \) by \( y \), we have

\[
x = y + \alpha \sin(\omega t). \tag{2}
\]

Substituting Eq. (2) into Eq. (1), equation of motion in relative coordinate system is

\[
\ddot{y} = \alpha \omega^2 \sin(\omega t), \quad -\frac{\nu}{2} < y < \frac{\nu}{2}. \tag{3}
\]

Integrating Eq. (3) and for the initial conditions \( y(t_0) = y_0 \) and \( \dot{y}(t_0) = \dot{y}_0 \), the displacement \( y \) and the velocity \( \dot{y} \), between impacts, are

\[
y(t) = y_0 + \alpha \sin(\omega t_0) - \alpha \sin(\omega t) + [\dot{y}_0 + \alpha \omega \cos(\omega t_0)](t - t_0) \tag{4}
\]

\[
\dot{y}(t) = \dot{y}_0 + \alpha \omega \cos(\omega t_0) - \alpha \omega \cos(\omega t). \tag{5}
\]

An impact occurs wherever \( y = \frac{\nu}{2} \) or \( -\frac{\nu}{2} \). After each impact, we apply into Eqs. (4) and (5) the new set of initial conditions (the Newton law of impact)

\[
t_0 = t, \quad y_0 = y, \quad \dot{y}_0 = -r \dot{y}, \tag{6}
\]

where \( r \) is a constant restitution coefficient.

Therefore, the dynamics of the impact-pair system is obtained from Eqs. (4), (5), and (6). In this case, the system depends on control parameters \( \nu, r, \alpha, \) and \( \omega \).

Since there is an analytical solution for the motion between impacts, we can obtain an impact map. First, we define the discrete variables \( y_n, \dot{y}_n, t_n \) as the displacement, the velocity, and the time (modulo \( 2\pi \)) collected just the \( n \)th impact. Substituting the Newton law of impact into Eqs. (4) and (5), we have a two-dimensional map that is given by:
\[ y_{n+1} = y_n + \alpha \sin(\omega t_n) - \alpha \sin(\omega t_{n+1}) + [-r y_n + \alpha \omega \cos(\omega t_n)](t_{n+1} - t_n) \]

\[ \dot{y}_{n+1} = -r y_n + \alpha \omega \cos(\omega t_n) - \alpha \omega \cos(\omega t_{n+1}), \]

where \( y_n = \nu/2 \) or \(-\nu/2\).

As mentioned before, this map is used in this work to evaluate the Lyapunov exponents [18]. Such exponents are computed through \( \lambda_i = \lim_{n \to \infty} (1/n) \ln |\Lambda_i(n)| \) (\( i = 1, 2 \)), where \( \Lambda_i(n) \) are the eigenvalues of the matrix \( A = J_1 \cdot J_2 \cdot \ldots \cdot J_n \) where \( J_n \) is the Jacobian matrix of the map (7), computed at time \( n \). For systems without analytical solutions between the impacts, the Lyapunov exponents can be calculated by using the method proposed by Jin and co-workers [22].

3. MULTISTABILITY WITH COMPLEX BASINS OF ATTRACTION

The dynamics was investigated using bifurcation diagrams, phase portraits, Lyapunov exponents, basins of attraction, uncertainty exponent, and parameter space diagrams. We fix the control parameters at \( \nu = 2.0 \) (length of the gap), \( \omega = 1.0 \) (excitation frequency) and vary the parameters \( r \) (restitution coefficient) and \( \alpha \) (excitation amplitude).

In order to obtain a representative example of the kind of dynamics generated by the impact-pair system, we use a bifurcation diagram for the velocity, \( \dot{y}_T \), versus the amplitude excitation, \( \alpha \). The dynamical variable \( \dot{y}_T \) is obtained from a stroboscopic map (Time-2\( \pi \)). To characterize the nature of the behavior observed, we calculate the Lyapunov exponents. If the largest Lyapunov exponent is positive the attractor is chaotic, if not the attractor is periodic.

In Fig. 2(a), we present a bifurcation diagram showing multiple coexisting attractors plotted in different colors. For example, we can note two period-1 orbits at \( \alpha = 3.2 \). In this case, these attractors are symmetric and appear due to the pitchfork bifurcation for \( \alpha \approx 3.0878 \). The Lyapunov exponents for attractors plotted in blue are shown in Fig. 2(b).

In Figs. 3(a) and 3(b), we show the phase portraits of the two symmetrical coexisting periodic attractors for \( \alpha = 3.2 \). For the same set of parameters, we identify more two possible solutions, namely two equilibrium points shown in Figs. 3(c) and 3(d).

The corresponding basins of attraction of the four possible solutions are depicted in Fig.
4(a). This figure are constructed using a grid of equally spaced 1000x1000 cells as set of initial conditions for velocity versus time (modulo 2π) with initial position fixed at \( y_0 = 0 \). The basins of the periodic attractors are plotted in blue and red, and the equilibrium points in white and green.

As can be seen, the structure of the basins is quite complex and the basin boundary between periodic attractors is convoluted and apparently fractal [Figs. 4(b) and (c)]. However, in contrast to fractal basin boundaries, we cannot observe an infinitely fine scaled structure with magnifications of the boundary region, as shown in Fig. 4(d). Therefore, the basin boundary here is composed of smooth curve with dimension one. In this case, the complexity of basins structure is generated by band accumulations [23, 24].

As a consequence of the complex basins of attraction observed, uncertainty in initial conditions leads to uncertainty in the final state. To evaluate the final state sensitivity we can calculate the uncertainty exponent, \( \beta \), which was proposed by Grebogi and his collaborators [25] and has been used to characterize fractal basin boundaries [10, 12].

To obtain the uncertainty exponent, we choose randomly a large number of set initial conditions \( A : (\tau_0, \dot{y}_0) \) for \( y_0 \) constant. We keep \( \dot{y}_0 \) constant and vary the other coordinate by a small amount \( \epsilon \). We also choose the slightly displaced initial conditions: \( B : (\tau_0 + \epsilon, \dot{y}_0) \) and \( C : (\tau_0 - \epsilon, \dot{y}_0) \). If the trajectory starting from initial condition \( A \) goes to one of the basins, and either one (or both) of the displaced initial conditions, \( B \) or \( C \), go to the other basin, we call \( A \) as an \( \epsilon \)-uncertain initial condition. By considering a large number of such points, we can estimate the fraction of \( \epsilon \)-uncertain points, \( f(\epsilon) \). This number scales with the uncertain radius as a power-law \( f(\epsilon) \sim \epsilon^\beta \), where \( \beta \) is called uncertainty exponent [23, 25].

For the basins of attraction shown in Fig. 4(c), we obtain the uncertainty exponent \( \beta = 0.490 \pm 0.001 \), as indicated in Fig. 5. In this case, if we want to gain a factor 2 in the ability to predict the final state of the system, it is necessary to increase the accuracy of initial condition by a factor approximately 4 \( (2^{1/0.49} \approx 4) \). Generally, the non-integer value of \( \beta \) is associated with fractal boundaries for dimension given by \( d = 2 - \beta \); here the non-integer value is attributed to accumulation boundary observed with dimension \( d = 1 \).
4. SELF-SIMILAR PERIODIC SETS IN PARAMETER SPACE

In order to obtain a further insight into the influence of the restitution coefficient \( r \) and amplitude excitation \( \alpha \) on the dynamics of the impact-pair system, we construct parameter space diagrams, depicted in Figs. 6(a) and 6(b). To obtain these diagrams, we use a grid of 1000x1000 cells with the initial conditions fixed at \((y_0, \dot{y}_0, \tau_0) = (0, 1, 0)\). For each point the largest Lyapunov exponent is calculated and plotted with the appropriately allocated color. Chaotic attractors \((\lambda_1 > 0)\) are plotted in blue and periodic attractors \((\lambda_1 < 0)\) according to a color scale ranging from minimum value in red and maximum in green. Zero Lyapunov exponents (bifurcation points) are plotted in blue.

To clarify how was constructed the parameter space diagram, we fix the restitution coefficient at \( r = 0.683 \) for Fig. 6(b) and vary the amplitude excitation \( \alpha \) determining bifurcation diagram and the corresponding largest Lyapunov exponents in Figs. 7(a) and 7(b), respectively. Comparing the Lyapunov exponents with parameter space diagram (Fig. 6(b)), we can note that red lines correspond with the local minimum of exponents and blue lines embedded in green region (periodic region) correspond with bifurcation points.

Examining the parameter space diagram [Fig. 6(a)], we note a main periodic structure existing embedded in chaotic region. Around this structure a vast quantity of self-similar periodic sets is found. For instance, we can see in Fig. 6(b) a very regular network of self-similar structures for a magnification of small box of Fig. 6(a). These self-similar structures have been observed before [26–30] and have been referred to as shrimps [31–33]. In our work, the shrimps organism themselves along a very specific direction in parameter space with a series of accumulations and fractality can be observed with successive magnifications of the parameter space.

To finalize, in Figs. 8(a) and 8(b) we depict two successive magnifications of the parameter space diagram [Fig. 6(a)]. In this case, we can observe a high concentration of periodic structures with different shapes [Fig. 8(a)]. These structures cross each other indicating coexistence of periodic attractors. In Fig. 8(b), we identify a periodic structure that appears abundantly in parameter space, for the system considered, consisting of three shrimp shapes connected. In addition, we again observe crossings between periodic sets.
5. CONCLUSIONS

In this paper, we considered the impact-pair system studying its dynamics by a means of numerical simulations. Initially, we discussed the coexistence of attractors with smooth basin boundary, but with complicated and evolved basins structure. According to the uncertainty exponent evaluated for a certain region in phase space, this type basins of attraction is associated with effect of obstruction on predictability of time-asymptotic final state (attractor). In other words, the uncertainty in initial conditions leads to uncertainty in the final state.

In the end, we explored the dynamics in the two-dimensional parameter space by using the largest Lyapunov exponents. We identified several periodic sets embedded in chaotic region. Some of them, known as shrimps, present self-similar structures and organize themselves along a very specific direction in parameter space with a series of accumulations. In addition, the periodic sets cross each other indicating a large quantity of coexisting periodic attractors.

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FIGURE CAPTIONS

FIGURE 1: Schematic view of an impact-pair system.

FIGURE 2: (a) Bifurcation diagram showing coexisting attractors. Velocity $\dot{y}_T$, for a stroboscopic map, as a function of the amplitude excitation, $\alpha$, for $r = 0.8$. (b) The Lyapunov exponents, $\lambda_{1,2}$, for the attractors plotted in blue.

FIGURE 3: Phase portrait of velocity versus displacement of two coexisting periodic attractors (a) and (b). Time histories of two equilibrium points (c) and (d). For the control parameters $\alpha = 3.2$ and $r = 0.8$.

FIGURE 4: Basins of attraction and successive magnifications for the parameter $\alpha = 3.2$ and $r = 0.8$, varying initial conditions $\dot{y}_0$ and $t_0$ with $y_0 = 0$. Basins in red and blue for the two coexisting attractors shown in Figs. 3(a) and 3(b), in green and white for the two equilibrium points shown in Figs. 3(c) and 3(d).

FIGURE 5: Uncertain fraction versus uncertainty radius $\epsilon$ for basins of attraction showed in Fig. 4(c). The solid curve is a linear regression fit with slope $\beta = 0.490 \pm 0.001$.

FIGURE 6: (a) Parameter space diagram with periodic structures. (b) Magnification of a portion of the previous figure.

FIGURE 7: Bifurcation diagram of velocity $\dot{y}_T$ as a function of the amplitude excitation $\alpha$ for $r = 0.683$. (b) The largest Lyapunov exponents, $\lambda_1$, for attractors of the bifurcation diagram.

FIGURE 8: (a) Parameter space diagram with periodic structures. (b) Magnification of a portion of the previous figure.
FIG. 2:
FIG. 3:
FIG. 4:
FIG. 5:
FIG. 6:
FIG. 7:
FIG. 8: