

Chapter 10

Fractal Structures in a Binary Schwarzschild Black Hole System

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In open Hamiltonian systems, the dynamics underlying chaotic scattering presents a number of fractal structures. One example is the deflection of light rays produced by a pair of supermassive (Schwarzschild) black holes. A light ray approaching this system can diverge to infinity, fall down into one of the black holes or can orbit around the black hole pair as periodic orbits. Using a two-dimensional area-preserving map proposed by Moura and Letelier, we investigated the escape basins and their fractal boundaries, using the corresponding basin and basin boundary entropies. We also varied the distance between the black holes, showing how it impacts in the complexity of the basin structure.

10.1. Introduction

Fractal structures appear often in chaotic systems, both dissipative and conservative ones [1]. In General Relativity, the geodesic motion in a curved space is a dynamical system of physical interest [2]. The presence of fractal structures in open chaotic systems is caused by the existence of an invariant nonattracting chaotic manifold, the so-called strange saddle [3], which influences the chaotic scattering of particles for example [4].

The chaoticity of geodesic motion is responsible for fractal distribution of the light ray scattering by a pair of black holes. This dynamics is described by the null geodesics equations [5]. One example is the Majumdar–Papapetrou binary black hole system [6, 7]. The investigation by Sanjuán and coworkers of fractal exit

basins in a Majumdar–Papapetrou binary black hole shows that the escape basin boundaries are not just fractal but also display the stronger Wada property: any boundary point belongs to the boundary of at least two other basins [5]. Moura and Letelier have proposed a two-dimensional map to describe the scattering of light rays by a system of two static Schwarzschild black holes [8].

We investigate the fractality of the escape basin by using a scattering map [8]. The escape basins are known to play an important role in astrophysical investigations, since they are actually the so-called shadows of a black hole [9]. Thus, we quantitatively investigate the fractal nature of the structure of escape basin by computing the basin entropy and the basin boundary entropy [10, 11].

This chapter is organized as follows. In Sec. 10.2, we introduce the basic formulas for the scattering of a light ray by a spherically symmetric black hole. In Sec. 10.3, we present the two-dimensional map that describes the light ray deflection due to the binary black hole system discussing some of its dynamic properties. In Sec. 10.4, we present some numerical results of the escape basin as we vary the distance between black holes. In Sec. 10.5, we characterize the fractality of the escape basins using the basin entropy and the basin boundary entropy. Finally, in last section we report our conclusions.

10.2. Basic Equations

Assuming the spacetime metric $g_{\mu\nu}$ with signature $(-, +, +, +)$ and using Einstein's summation convention for repeated indexes, we define $c = G = 1$. In a curved spacetime, the light rays follow geodesics given by

$$g_{\mu\nu}dx^\mu dx^\nu = 0, \quad (10.1)$$

where we consider a metric for a symmetrically spherical and static spacetime with length element

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (10.2)$$

where $A(r)$, $B(r)$ and $C(r)$ define the metric produced by the black hole.

For a photon approaching the black hole from infinity with impact parameter b , the substitution of (10.2) into (10.1) gives the following general equation for the geodesics

$$\frac{A(r)B(r)}{[C(r)]^2} \left(\frac{dr}{d\phi} \right)^2 + \frac{1}{[U(r)]^2} = \frac{1}{b^2}, \quad (10.3)$$

where $[U(r)]^{-2} = A(r)/C(r)$. After approaching the black hole with a minimum distance r_0 the photon is deflected and emerges out in the other direction. From (10.3) this distance is called the critical impact parameter

$$b_c = \sqrt{\frac{C(r_0)}{A(r_0)}} \equiv \sqrt{\frac{C_0}{A_0}}. \quad (10.4)$$

On substituting (10.4) back into (10.3) it turns out that the angle of deflection is $\alpha = I(r_0) - \pi$, where

$$I(r_0) = \int_{r_0}^{\infty} dr \sqrt{\frac{B(r)}{C(r)}} \left(\frac{A_0}{A(r)} \frac{C(r)}{C_0} \right)^{-1/2}. \quad (10.5)$$

In the limit of strong gravitational fields, we can expand (10.5) so as to obtain

$$\begin{aligned} \alpha(r_0) = & -a \ln \left(\frac{r_0}{r_m} \right) + c + o(r_0 - r_m) = -\bar{a} \ln \left(\frac{b}{b_m} - 1 \right) \\ & + \bar{c} + o(b - b_m), \end{aligned} \quad (10.6)$$

where a , \bar{a} , c and \bar{c} depend on the functions A , B and C , evaluated at the photosphere radius r_m . Similarly to (10.4) we have $b_m = \sqrt{C(r_m)/A(r_m)}$.

We shall consider Schwarzschild black holes, such that these functions are given, in a suitable system of units [12]

$$A(r) = 1 - \frac{2M}{r}, \quad (10.7)$$

$$B(r) = \left(1 - \frac{2M}{r} \right)^{-1} = \frac{1}{A(r)}, \quad (10.8)$$

$$C(r) = r^2. \quad (10.9)$$

The Schwarzschild metric has an event horizon, given by the radii where it diverges, corresponding to $r = 2M$, without loss of

generality we will rewrite

$$R = \frac{r}{2M}. \quad (10.10)$$

Using (10.7)–(10.9) and computing the extreme of the effective potential results that the radius of the photosphere is $R_m = 3/2$.

From these results, Bozza [13] shows that the coefficients of the expansion for the deflection angle (10.6) are given by

$$\bar{a} = 1, \quad (10.11)$$

$$\bar{c} = -\pi + 0.9496 + \ln 6 \approx -0.4002, \quad (10.12)$$

$$b_m = \frac{3\sqrt{3}}{2} \approx 2.5981. \quad (10.13)$$

The scattering of a light ray by a single black hole can now be understood in terms of the possible values of the corresponding impact parameter. A light ray comes from infinity and approaches the black hole with impact parameter b and whose direction makes an angle ϕ with a reference axis. If $b < b_m$, the light ray falls into the black hole and disappears. On the other hand, for this light ray not to escape back to infinity, it is necessary that its impact parameter b be such that the scattering angle α is at least π . Thus we impose $b > b_{\text{esc}}$, where $\alpha(b_{\text{esc}}) = \pi$. Using (10.6), this means that

$$b_{\text{esc}} = b_m \left[\exp \left(\frac{\bar{c} - \pi}{\bar{a}} \right) + 1 \right] \approx 2.67332. \quad (10.14)$$

As a result, for a light ray deflected by a single black hole not to escape to infinity or to collide with a black hole, the impact parameter must belong to the narrow interval $b_m < b < b_{\text{esc}}$.

10.3. The Scattering Map

When working with a system of two black holes in orbit of each other, the map has no exact solution of field equations (unlike the case of a single black hole treated in the previous section). But the nonlinear interaction between their gravitational fields can be neglected if the distance D between the black holes is much higher than their Schwarzschild radius $R = 1$.

In this case, we consider the deflection of light from each black hole in a separate way using the expressions previously found for the Schwarzschild metric. In other words, in the light scattering by a given black hole, the effect of the other black hole is neglected. Similar approximations for the Schwarzschild metric are discussed by de Moura [8]. In particular, they may not hold if we consider the scattering of massive test particles.

The axial symmetry axis is the line connecting the two black holes. Assuming that the light rays have zero angular momentum in this direction, as a result, the light rays are constrained to move in the plane containing the two black holes. Figure 10.1 shows the basic geometry involved in the light scattering by the system. Coming from infinity, a light ray approaches the first black hole with impact parameter b , the direction makes an angle ϕ with the axial symmetry line. So that this light ray does not escape back to infinity it is necessary that $b < b_{\text{esc}}$, where b_{esc} is given by (10.14). As we do not want the light ray to fall to the first black hole as soon as they meet, we define the impact parameter as $b > b_m$. As the light ray is not deflected to infinity by the first black hole, make it goes to the other black hole and is again deflected. Again, if not deflected to infinity it returns to the vicinity of the first black hole, *ad infinitum*.

For convenience, we define discrete variables (b_n, ϕ_n) , the impact parameter and escape angle, in reference to the axis symmetry line

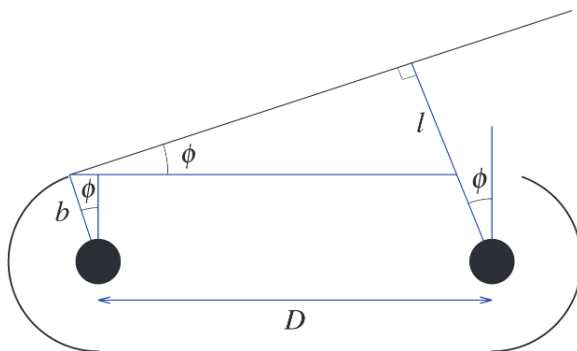


Fig. 10.1. Schematic figure for the trajectory of a light ray in the two black hole system.

in the neighborhood of the n th scattering. Odd (even) values of n correspond to the second (first) black hole. Using such definition, the differential equations for light scattering reduce to a discrete map,

$$b_{n+1} = b_n + D\phi_n, \quad (10.15)$$

$$\phi_{n+1} = \pi + \phi_n - \alpha(b_{n+1}) \pmod{2\pi}, \quad (10.16)$$

which is called scattering map by Moura and Letelier [8]. Some sign conventions are essential: positive values of b imply that the light ray goes from black hole 1 to 2 (from “right” to “left” in Fig. 10.1), and negative values otherwise; whereas positive values of ϕ correspond to counterclockwise rotations.

The Jacobian matrix of this transformation is given by

$$\mathbf{J} = \begin{pmatrix} 1 & D \cos \phi_n \\ -\alpha'(b_{n+1}) & 1 - D\alpha'(b_{n+1}) \cos \phi_n \end{pmatrix}, \quad (10.17)$$

whose determinant is equal to the unity, hence the scattering map (10.15)–(10.16) is a twist and area-preserving mapping, corresponding to the following continuous-time Hamiltonian:

$$H(b, \phi, n) = \pi b - \int^b db' \alpha(b') + D\delta_1(n) \cos \phi, \quad (10.18)$$

where

$$\delta_1(n) = \sum_{m=-\infty}^{\infty} \delta(n-m) = 1 + 2 \sum_{q=1}^{\infty} \cos(2\pi qn), \quad (10.19)$$

is a periodic delta function (“Dirac comb”).

Fixed points of the scattering map are $\phi_{1,2}^* = 0, \pi$ and $b_{1,2}^* = b_{\text{esc}}$. The eigenvalues of the Jacobian matrix (10.17) at these points are

$$\lambda_{1,2} = \frac{\zeta}{2} \pm i \sqrt{1 - \left(\frac{\zeta}{2}\right)^2}, \quad (10.20)$$

where

$$\zeta = 2 - D\alpha'(b_{n+1}) \cos \phi_n \quad (10.21)$$

is the trace of Jacobian matrix.

These fixed points are stable provided that $|\zeta| < 2$. It turns out that the point $(b_{\text{esc}}, 0)$ is stable if $0 < D\alpha'(b_{\text{esc}}) < 4$, whereas the

other fixed point (b_{esc}, π) is stable if $-4 < D\alpha'(b_{\text{esc}}) < 0$. In both cases, the light ray trajectory is such that $b_{n+1} = -b_n$. Considering $D > 0$ and $\alpha'(b_{\text{esc}}) < 0$ the fixed point at $\phi = 0$ is always unstable (a hyperbolic saddle). If, further, we have that $|\alpha'(b_{\text{esc}})| < (4/D)$, the other fixed point at $\phi = \pi$ is an elliptic center.

The Hamiltonian (10.18) is generally nonintegrable for $D \neq 0$. However, if the light deflection by one black hole has to be independent on the existence of the another black hole, then we must assume that D is typically a large number. In such an instance, both fixed points are unstable and we expect a sizeable area-filling chaotic orbit in the phase space (b, ϕ) .

10.4. Escape Basins

The system proposed of two black holes is an example of open dynamical system, which is a system where trajectories (light rays) eventually escape from a given phase space region. If there are different ways by which trajectories can escape, then it is an applicable problem to identify the sets of initial conditions that causes trajectories to escape through a given exit. This array is called the escape basin, which corresponds to that exit [4]. In the case of two or more exits, we can identify the boundary that divides those escape basins, termed escape basin boundary. It is long known that conservative dynamical systems with chaotic dynamics have fractal escape basins and fractal escape basin boundaries [1].

With the desire to obtain the plot of the escape basins corresponding to the scattering map (10.15)–(10.16), we designated a set of initial conditions (b_0, ϕ_0) and iterated them to encounter to which basin they belong. We isolate the phase space region $\Omega = \{0 \leq \phi_0 \leq 2\pi, b_m < b_0 < b_{\text{esc}}\}$ in a extensive number of points and iterate the map (10.15)–(10.16) for each of these initial conditions, saving the final outcome for each point. Succeeding a number of map iterations, depending on its initial conditions, a light ray may fall into one black hole (**A**), into the other black hole (**B**), or escapes in the direction of infinity (**C**). All outcomes can be treated as exits since we stop iterating the map once a light ray falls into a black

hole. Accordingly, we denote the corresponding escape basins to be $\mathcal{B}(\mathbf{A})$, $\mathcal{B}(\mathbf{B})$ and $\mathcal{B}(\mathbf{C})$.

To do this analysis, we use a range of values for D from $D = 5$ until $D = 20$, which is a value sufficient to ensure that each black hole deflects light rays in an independent fashion. An orbit falls into a black hole whenever $b_n < b_m$ for a given escape time $n = \bar{n} < 10^4$. If \bar{n} is even (odd) we know that the light ray falls into black hole \mathbf{A} (\mathbf{B}) and the corresponding initial condition is painted red (blue). If the orbit goes to infinity ($b_n > b_{\text{esc}}$) for a given $n = \bar{n} < 10^4$ the corresponding initial condition is painted green. The red, blue and green regions are thus numerical approximation for the exit basins $\mathcal{B}(\mathbf{A})$, $\mathcal{B}(\mathbf{B})$ and $\mathcal{B}(\mathbf{C})$, respectively. There is a measured set of unstable periodic orbits which never escape and, since $D < \infty$, we cannot rule out orbits within very tiny periodic islands which do not escape at all, but whose effect in the exit basins would be negligible.

Since the labeling of black holes \mathbf{A} and \mathbf{B} is immaterial, their escape basins would be symmetric, i.e., they must have the same size. On the other hand, it is expected that the dominant basin is that of infinity (\mathbf{C}). But the analyses require further magnifications as sequences of exits cannot be observed. This characterization is also possible by defining a function $g(b)$ such that $g(b) = 1$, if the orbit falls into black hole \mathbf{A} , $g(b) = -1$, if it falls into \mathbf{B} , and $g(b) = 0$ if the orbit escapes to infinity [8].

In order to investigate the escape basin boundary, we analyze two regions in the vicinity of the points with $\phi_0 = 0$ and $\phi_0 = \pi$. For both regions, we divide the impact parameter interval $b_m < b < b_{\text{esc}}$ into 10^6 points. The corresponding escape basins are plotted in Figs. 10.2(a) and 10.2(d), respectively, as a horizontal bar with green, red and blue stripes. We also plotted the corresponding function $g(b)$ below the bars. In Figs. 10.2(b) and 10.2(e), we show magnifications of two intervals of Figs. 10.2(a) and 10.2(d), respectively, and Figs. 10.2(c) and 10.2(f) are further magnifications. These zooms clearly show that there are regions for which there are pieces of the three escape basins in all scales. This self-similarity is a signature of the fractality of the basins as well as of its basin boundary.

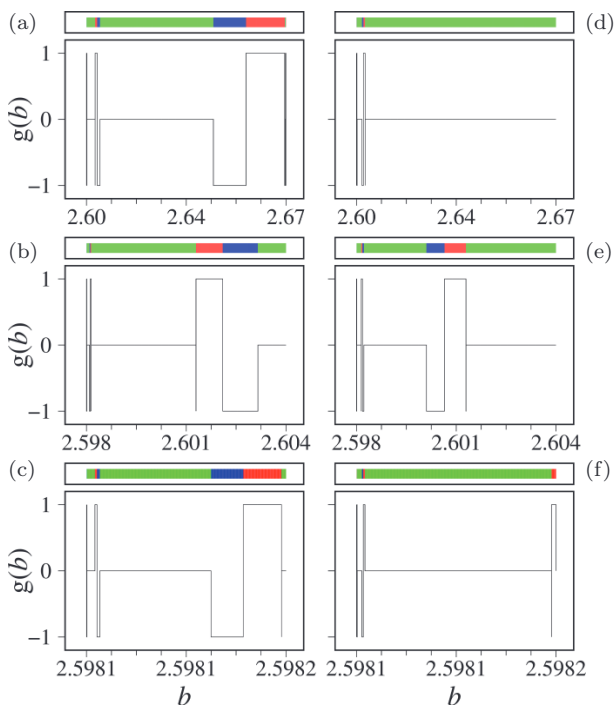


Fig. 10.2. For $D = 12, 5$, the horizontal bars represent the escape basins as a function of the impact parameter b for $\phi_0 = 0$ in (a), with magnifications in (b) and (c); and $\phi_0 = \pi$ in (d), with magnifications in (e) and (f). We plot, below the horizontal bars, the corresponding values of the function $g(b)$ (see text for details).

A cursory inspection of Fig. 10.2 suggests that the escape basin boundary is self-similar and has a fractal structure that depends of b . Actually, the escape basin boundary cannot be completely fractal, since there are smooth parts of it corresponding to regions in which the function $g(b)$ has the same value for different ranges of b .

Since some regions of different escape basins are intertwined in arbitrarily fine scales, it is extremely difficult, if not impossible, to predict the final outcome of a light ray, given its initial condition being always known up to a given uncertainty. One of the observable consequences of the existence of fractal structures in phase space is

final-state sensitivity, i.e., small uncertainties in the initial conditions may lead to large uncertainties with respect to the future behavior of the system [14].

10.5. Basin Entropies

We quantify the degree of uncertainty due to the fractality of the escape basin boundary by the determination of the so-called basin entropy and basin boundary entropy, a tool recently developed by Miguel Sanjuán and collaborators [5, 10]. Let us suppose that an open Hamiltonian system, like the scattering map, has N_E exits, by which typical trajectories can escape, with the corresponding escape basins. We cover the phase space region with a grid of N boxes of size ε , and consider the probability p_{ij} for the trajectory beginning in the j th box to escape through the i th exit, where $i = 1, 2, \dots, N_E$ and $j = 1, 2, \dots, m_i$, where $m_i \in [1, N_E]$ is the number of exits available to trajectories originating from the i th box.

On supposing that the trajectories beginning in a box are statistically independent, Daza *et al.* defined the entropy of the i th box as

$$S_i = - \sum_{j=1}^{m_i} p_{ij} \log(p_{ij}), \quad (10.22)$$

in such a way that the entropy of the grid is obtained by summing over all N boxes, $S = \sum_{i=1}^N S_i$. The basin entropy is defined as $S_b = S/N$.

The basin entropy quantifies the degree of the uncertainty associated with the escape basin, and varies from 0, in the case of a unique exit, to $\log N_E$, if the escape basins are so completely intertwined that we may consider them as equiprobable. If we restrict the computation of the basin entropy to those N_b boxes covering the escape basin boundaries, then we have the basin boundary entropy $S_{bb} = S/N_b$, which quantifies the uncertainty related only to the boundary.

In order to compute the basin entropy and basin boundary entropy of the escape basin of Fig. 10.2, we analyze the regions

$\phi_0 = 0$ and $\phi_0 = \pi$, separately. For both regions, we consider boxes with 4, 5, 9, 10 initial conditions per box. For each initial condition, we computed a maximum number of 10^4 iterations of the map and exclude from the statistics those initial conditions leading to orbits that do not escape during this maximum iteration time. Then, we compute the probabilities (10.23) of obtain the number of points into each box n_A , n_B and n_C corresponding to the exits **A**, **B** and **C**, respectively.

$$p_A = \frac{n_A}{n_A + n_B + n_C}, \quad p_B = \frac{n_B}{n_A + n_B + n_C}, \quad p_C = \frac{n_C}{n_A + n_B + n_C}, \quad (10.23)$$

and the entropy for each box is

$$S = -p_A \log p_A - p_B \log p_B - p_C \log p_C. \quad (10.24)$$

The entropy related to the each region of the escape basin is obtained by summing the contribution of each box $S = \sum_{i=1}^N S_i$. Thus, the basin entropy results by normalizing this value $S_b = S/N$. In limit situations, if we have a single exit, the corresponding probability is equal to the unity ($p_A = 1$), hence $S_b = 0$, i.e., no uncertainty at all. The opposite situation consists in completely randomized basins with N equiprobable escapes, which result in $S_b = \log N$ as the upper bound of the basin entropy.

We obtain the so-called boundary basin entropy $S_{bb} = S/N_b$ limiting the computation of the basin entropy only to the boxes containing points of the boundary (N_b). Thus, the basin boundary entropy S_{bb} measures the complexity of the basin boundary. In the case of two basins with smooth boundaries, the number of boxes in the boundary is negligible for the computation of the boundary basin entropy S_{bb} , since there are many more boxes with just one basin. Thus, the maximum possible value of S_{bb} that a smooth boundary can have is $\ln 2$, which would a pathological case where every box in the boundary contains equal proportions. Therefore, if $S_{bb} > \ln 2$, the basin boundary is said to be fractal.

We plot in Fig. 10.3 the values of the basin entropy S_b and the basin boundary entropy S_{bb} of the escape basin considering the two regions (a) $\phi = 0$ and (b) $\phi = \pi$. The variable parameter is the

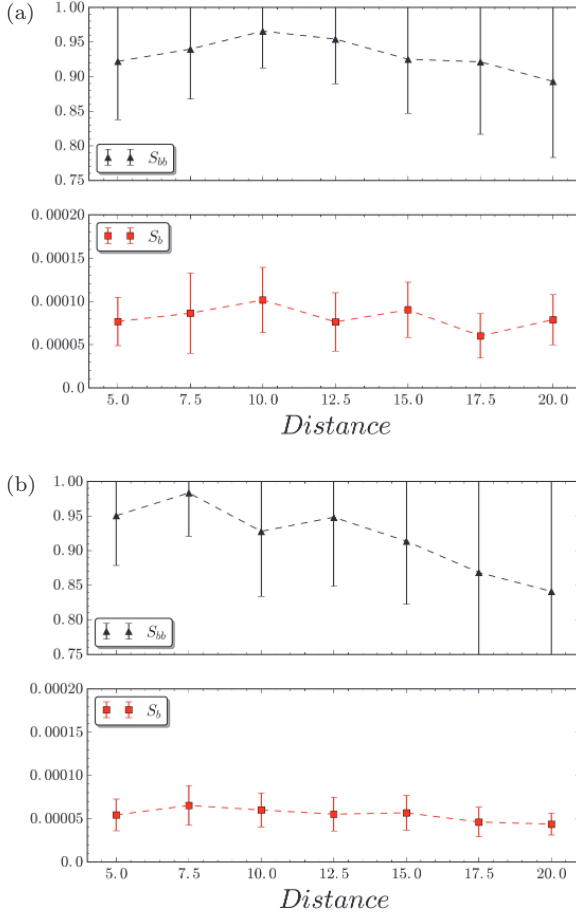


Fig. 10.3. Basin entropy S_b and basin boundary entropy S_{bb} as a function of distance between the black holes D for (a) $\phi = 0$ and (b) $\phi = \pi$.

distance between the black holes D . For the two cases of ϕ the value of basin entropy S_b is low, in fact most boxes contains initial conditions that escape through a single exit. However, the entropy of the boundary of basin S_{bb} obeys the inequality $S_{bb} > \ln 2$, i.e., the basin boundary is fractal.

As a general trend, the degree of complexity of the basin structure and its boundary decreases with D . We can observe in (a) that the entropies increase as D goes from 5 to 10 and suffer a decrease

afterwards. The same happens in (b) but the maximum entropy value occurs in $D = 7.5$ and decreases for higher values of D . The decrease of entropy means that the basins become progressively less mixed and involved as the distance between the black holes increases.

10.6. Conclusion

An outstanding example of open Hamiltonian systems is the scattering of light rays by a binary black hole system. But as an open system, the chaotic motion is typically transient, the destiny of an incident light ray is either to escape to infinity or fall into one of the black holes. In these cases, we seek to obtain the respective exit basin, or the set of initial conditions that attend to a given outcome. The exit basins and their common boundary are fractal structures, essentially due to the presence of a nonattractive invariant set called chaotic saddle.

We considered the light ray scattering by a binary black hole system from the angle of an open Hamiltonian systems, focusing on the fractal structures present in the chaotic dynamics. The equations of general relativity for a light ray in the gravitational field of a binary Schwarzschild black hole system were integrated approximately to gather a discrete-time map, which exhibits chaotic dynamics for a wide range of its parameters, with the most important being the distance between the black holes. The chaotic dynamics here are transient, though, for the light rays can either escape to infinity or fall into one of the black holes.

The basin entropy, which is basically the information entropy related to the probability of going to a given basin, has been found to vary according to the distance used to represent the exit basins. The basin boundary entropy only takes into account the intervals containing basin boundary.

This work can be viewed as a connection to the research on black hole shadows as we can see in [9, 15], the shadow is a direct result of gravitational lensing as the black holes block some of the light rays that converge to it. As we know, the study of black hole shadows is important on the visualization and classification of black holes.

The shadow that it produces is an exit basin, so in this work we can use the exit basin we found to see the black hole shadow if necessary.

In summary, we have studied fractal structures that appear in the chaotic motion of a light ray under the gravitational field of two Schwarzschild black holes, a problem which is classically nonintegrable and has a nonattractive chaotic invariant set responsible for chaotic transient dynamics. Such fractal structures are responsible for various signatures of the so-called chaotic scattering, which is a phenomenon ubiquitous in nonintegrable open Hamiltonian systems.

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References

- [1] J. Aguirre, R. L. Viana and M. A. F. Sanjuán, Fractal structures in nonlinear dynamics, *Rev. Mod. Phys.* **81** (2009) 333–386; doi: 10.1103/RevModPhys.81.333.
- [2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972).
- [3] A. Péntek, Z. Toroczkai, T. Tél, C. Grebogi and J. A. Yorke, Fractal boundaries in open hydrodynamical flows: Signatures of chaotic saddles, *Phys. Rev. E* **51** (1995) 4076–4088; doi: 10.1103/PhysRevE.51.4076.
- [4] S. Bleher, C. Grebogi, E. Ott and R. Brown, Fractal boundaries for exit in hamiltonian dynamics, *Phys. Rev. A* **38** (1988) 930–938; doi: 10.1103/PhysRevA.38.930.
- [5] A. Daza, J. O. Shipley, S. R. Dolan and M. A. F. Sanjuán, Wada structures in a binary black hole system, *Phys. Rev. D* **98** (2018) 084050; doi: 10.1103/PhysRevD.98.084050.
- [6] S. D. Majumdar, A class of exact solutions of Einstein’s field equations, *Phys. Rev.* **72** (1947) 390–398; doi: 10.1103/PhysRev.72.390.
- [7] A. Papapetrou, A static solution of the equations of the gravitational field for an arbitrary charge-distribution, *Proc. Royal Irish Acad. Sect. A: Math. Phys. Sci.* **51** (1945) 191–204.

- [8] A. P. S. de Moura and P. S. Letelier, Scattering map for two black holes, *Phys. Rev. E* **62** (2000) 4784–4791; doi: 10.1103/PhysRevE.62.4784.
- [9] P. V. P. Cunha and C. A. R. Herdeiro, Shadows and strong gravitational lensing: A brief review, *Gen. Relativ. Grav.* **50**(4) (2018) 42; doi: 10.1007/s10714-018-2361-9.
- [10] A. Daza, A. Wagemakers, B. Georgeot, D. Guéry-Odelin and M. A. F. Sanjuán, Basin entropy: A new tool to analyze uncertainty in dynamical systems, *Sci. Rep.* **6**(1) (2016) 31416; doi: 10.1038/srep31416.
- [11] A. Daza, B. Georgeot, D. Guéry-Odelin, A. Wagemakers and M. A. F. Sanjuán, Chaotic dynamics and fractal structures in experiments with cold atoms, *Phys. Rev. A* **95** (2017) 013629; doi: 10.1103/PhysRevA.95.013629.
- [12] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).
- [13] V. Bozza, Gravitational lensing in the strong field limit, *Phys. Rev. D* **66**, (2002) 103001; doi: 10.1103/PhysRevD.66.103001.
- [14] S. W. McDonald, C. Grebogi, E. Ott and J. A. Yorke, Fractal basin boundaries, *Physica D* **17**(2) (1985) 125–153; doi: doi.org/10.1016/0167-2789(85)90001-6.
- [15] A. Daza, J. O. Shipley, S. R. Dolan and M. A. F. Sanjuán, Wada structures in a binary black hole system, *Phys. Rev. D* **98** (2018) 084050; doi: 10.1103/PhysRevD.98.084050.