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ABSTRACT

Using a method developed by Clemente, it is possible to obtain anisotropic magnetohydrodynamic equilibrium in axially symmetric systems, from a previously known solution of the Grad-Schlüter-Shafranov equation. We generalize this method to symmetric systems described by orthogonal as well as nonorthogonal systems of coordinates. Two examples are presented in cylindrical and spherical geometries, for which we give an exact analytic solution of the anisotropic MHD equilibrium.

I. INTRODUCTION

When there exists a strong magnetic field acting upon a fusion plasma, it may be adequate to replace the isotropic plasma pressure by an anisotropic expression which distinguishes between pressures along and across the magnetic field local direction. Such a distinction can arise, for example, when auxiliary heating methods like neutral-beam injection are used. In a seminal paper, Chew, Goldberger, and Low (CGL)[1] have suggested the use of an anisotropic form for the pressure tensor. The fluid equations obtained with such an anisotropic tensor are now called double adiabatic equations.[1,3]

The ideal magnetohydrodynamic (MHD) equilibrium equation with the anisotropic pressure tensor in the CGL form has been written by the first time by Mercier and Cotsaftis, who also studied the MHD stability of the resulting equilibrium configurations.[4] Sestero and Taroni have obtained numerical solutions for toroidal equilibrium with anisotropic pressure in the presence of a vacuum toroidal field.[5] The dynamics and stability of the axisymmetric and anisotropic plasma-vacuum system have been studied by Fielding and Haas.[6] Numerical investigations of anisotropic equilibria within the framework of a CGL pressure tensor have been carried out by Cooper and co-workers.[7,8] Approximate solutions of the anisotropic equilibrium equations have also been obtained.[9,10]

Clemente[11] obtained a method to construct an infinite number of axisymmetric anisotropic MHD equilibria by transforming them to the Grad-Schlüter-Shafranov (GSS) equation describing equilibria with isotropic pressure.[12–14] This method enabled the obtention of the first analytical solutions for the anisotropic equilibrium[11] from two known solutions of the GSS equation, namely a Hill vortex model for field reversed configuration and the Maschke-Hernegger solution in cylindrical coordinates.[15,16]

The power of the Clemente method can only be fully harnessed, however, if the anisotropic equilibrium equation is expressed in a coordinate system with one ignorable coordinate to play the role of axisymmetry. In this work, we write down such an equation by considering suitable representations for quantities like the transversal flux function and current function, following the steps laid down by previous work on the isotropic MHD equation for symmetric systems.[17,18] We apply the Clemente method to show how solutions in the anisotropic case can arise from transformations of the isotropic case, whenever the latter has previously known solutions.

We consider in detail two applications of the anisotropic equation: the first, in cylindrical coordinates, is the Solovev solution,[19] which is widely used as a benchmark for numerical codes of equilibrium calculation.[20] The other one, in spherical coordinates, uses the solution proposed by Morikawa to the problem of a plasma sphere surrounded by a force-free medium and a spherical conducting shell.[21] In both cases, we obtain analytical solutions for the anisotropic equation that enable us to calculate the radial profiles of the parallel and perpendicular pressures with respect to the local magnetic field direction.

This paper is organized as follows: in Sec. II, we consider the basic equations and the flux-based quantities to be used in this work. In Sec. III, we present a detailed derivation of the MHD equilibrium equation with anisotropic pressure given by the CGL model. The Clemente method for obtaining solutions of the anisotropic equation is shown in Sec. IV. Section V contains a discussion of the equilibrium equation in cylindrical coordinates, whereas Sec. VI considers spherical coordinates. Our Conclusions are presented in Sec. VII.

II. BASIC EQUATIONS

We start from the ideal MHD equations (SI units, where $c = 1/\sqrt{\mu_0 \varepsilon_0}$, are used)[23,24]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = -\nabla \cdot \mathsf{T} + \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \tag{4}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{5}$$

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \tag{6}$$

where ρ, s, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$ represent the plasma density, specific entropy, current density, electric field, and magnetic induction, respectively. Here, T represents the pressure tensor which, for an isotropic plasma, is spherical $\mathsf{T} = p\mathsf{I}$, where p is a scalar pressure and I is the identity tensor.

The set of ideal MHD equations describing static ($\mathbf{v} = 0$) equilibrium (magnetohydrostatics) is

$$\nabla \cdot \mathsf{T} = \mathbf{J} \times \mathbf{B}, \tag{7}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{8}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{9}$$

A general description of axisymmetric MHD equilibria is obtained per using curvilinear contravariant coordinates (x^1, x^2, x^3) for a system described by the contravariant metric tensor $g^{ij} = \hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}^j$, where $\hat{\mathbf{e}}^i = \nabla x^i$ are contravariant basis vectors. Given an arbitrary vector $\mathbf{a}$ with contravariant components (a^1, a^2, a^3), its covariant components (a_1, a_2, a_3) can be obtained by application of the covariant metric tensor $g_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$, where $\hat{\mathbf{e}}_i = \partial \mathbf{r}/\partial x^i$ are covariant basis vectors and $\mathbf{r}$ is the position vector.[25–27]

We suppose the existence of an ignorable coordinate $0 \le x^3 \le L$. The magnetic axis is a degenerate flux surface given by the x^3-coordinate curve, for which $x^1 = a$.[17,18] If we consider a coordinate surface $x^2 =$ const. and call S_2 the annulus bounded by the magnetic axis and a coordinate curve x^3, the transversal flux function can be defined as the magnetic flux through S_2 per unit length in the x^3 direction (Fig. 1)[18]

$$\Psi(x^1, x^2) = \frac{1}{L}\int_a^{x^1} dx'^1 \int_0^L dx^3 \sqrt{g} B^2, \tag{10}$$

where $g = \det(g_{ij})$. From Eq. (9) and using the symmetry with respect to x^3, there results

$$B^1(x^1, x^2) = -\frac{1}{\sqrt{g}}\frac{\partial \Psi}{\partial x^2}, \quad B^2(x^1, x^2) = \frac{1}{\sqrt{g}}\frac{\partial \Psi}{\partial x^1}, \tag{11}$$

such that we can write a representation for the magnetic field

$$\mathbf{B} = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla \Psi + B_3 \frac{\hat{\mathbf{e}}_3}{g_{33}}, \tag{12}$$

where $\hat{\mathbf{e}}_3$ is the covariant basis vector.

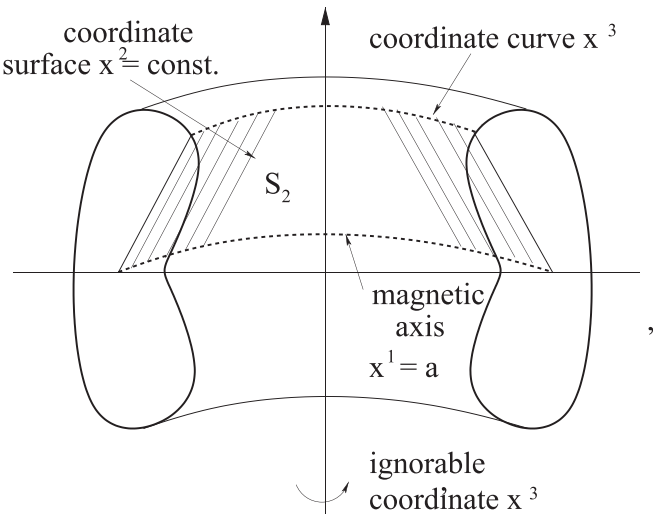

FIG. 1. Schematic figure showing some coordinate surfaces and curves in an axisymmetric system.

Taking the dot product of (12) with $\nabla\Psi$, we have $\mathbf{B} \cdot \nabla\Psi = 0$, i.e., $\Psi(x^1, x^2)$ has a constant value on a given flux surface. The divergence of Ampère's law (8) yields $\nabla \cdot \mathbf{J} = 0$ so that, in analogy with the previous case, we can write a representation for the current density, as follows:

$$\mathbf{J} = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla I + J_3 \frac{\hat{\mathbf{e}}_3}{g_{33}}, \tag{13}$$

where we define a transverse current flux $I(x^1, x^2)$ in terms of the total current flowing through the surface S_2 per unit length

$$I(x^1, x^2) = I_{axis} + \frac{1}{L}\int_a^{x^1} dx'^1 \int_0^L dx^3 \sqrt{g} J^2, \tag{14}$$

where $I_{axis} = I(x^1 = a, x^2)$, in such a way that

$$J^1 = -\frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^2}, \quad J^2 = \frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^1}. \tag{15}$$

Writing the magnetic field as $\mathbf{B} = \nabla \times \mathbf{A}$ in (10) yields $\Psi = -A_3$ so that, similarly with what has been done with Ampère's law (8), we obtain $B_3(x^1, x^2) = -\mu_0 I(x^1, x^2)$.[18] The representation (12) can thus be rewritten as

$$\mathbf{B} = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla \Psi - \mu_0 I \frac{\hat{\mathbf{e}}_3}{g_{33}}, \tag{16}$$

the square modulus of which is thus

$$|B|^2 = \frac{1}{g_{33}}\left\{|\nabla \Psi|^2 + \mu_0^2 I^2\right\}. \tag{17}$$

Substituting (16) in Eq. (8) gives, after dotting with $\hat{\mathbf{e}}_3$, the current density component along the symmetric direction as

$$J_3 = \frac{1}{\mu_0}\Delta^* \Psi - I\mathcal{D}, \tag{18}$$

in which we defined a generalized Shafranov operator[17]

$$\Delta^*\Psi = \frac{g_{33}}{\sqrt{g}}\left\{\frac{\partial}{\partial x^1}\left[\frac{\sqrt{g}}{g_{33}}\left(g^{11}\frac{\partial\Psi}{\partial x^1}+g^{12}\frac{\partial\Psi}{\partial x^2}\right)\right] + \frac{\partial}{\partial x^2}\left[\frac{\sqrt{g}}{g_{33}}\left(g^{12}\frac{\partial\Psi}{\partial x^1}+g^{22}\frac{\partial\Psi}{\partial x^2}\right)\right]\right\} \tag{19}$$

and a factor which is nonzero only for nonorthogonal coordinate systems

$$\mathcal{D} = \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{23}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{13}}{g_{33}}\right)\right]. \tag{20}$$

Substituting (16) and (13) in the Lorentz force term, we have

$$\mathbf{J}\times\mathbf{B} = -\frac{1}{g_{33}}(J_3\nabla\Psi + \mu_0 I\nabla I). \tag{21}$$

Taking the dot product with $\mathbf{B}$ yields $\mathbf{B}\cdot\nabla I = 0$, i.e., $I = I(\Psi)$ is also constant on a given flux surface, so that $\nabla I = I'\nabla\Psi$ and

$$\mathbf{J}\times\mathbf{B} = -\frac{1}{g_{33}}\left[\frac{1}{\mu_0}\Delta^*\Psi - I\mathcal{D} + \frac{1}{2}\mu_0(I^2)'\right]\nabla\Psi, \tag{22}$$

where the prime denotes differentiation with respect to Ψ.

III. ANISOTROPIC PRESSURE UNDER MERCIER-COTSAFTIS HYPOTHESES

If there is a strong magnetic field, the plasma responses are generally different at directions parallel and perpendicular to $\mathbf{B}$, so that we can use the Chew-Goldberger-Low expression[1,2]

$$\mathsf{T} = p_\perp \mathsf{I} + (p_\parallel - p_\perp)\frac{\mathbf{BB}}{B^2} = p_\perp\mathsf{I} + \frac{1}{\mu_0}\sigma_-\mathbf{BB}, \tag{23}$$

where $p_\parallel$ and $p_\perp$ are the pressures along and across the magnetic field and we define

$$\sigma_- = \frac{p_\parallel - p_\perp}{|B|^2/\mu_0}. \tag{24}$$

From the condition $\nabla\cdot\mathbf{B} = 0$, it turns out that

$$\nabla\cdot\mathsf{T} = \nabla p_\perp + \frac{1}{\mu_0}\mathbf{B}(\mathbf{B}\cdot\nabla\sigma_-) + \frac{1}{\mu_0}\sigma_-(\mathbf{B}\cdot\nabla)\mathbf{B}. \tag{25}$$

It is also convenient to define a mean pressure

$$\bar{p} = \frac{1}{2}(p_\parallel + p_\perp). \tag{26}$$

According Mercier and Cotsaftis, we shall use the following hypotheses[4]

- σ_- depends only on $\Psi(x^1, x^2)$,
- $\bar{p}$ depends only on $\Psi(x^1, x^2)$,

such that $\nabla\sigma_- = (\sigma_-)'\nabla\Psi$ and $\nabla\bar{p} = (\bar{p})'\nabla\Psi$. As a result

$$\mathbf{B}\cdot\nabla\sigma_- = (\mathbf{B}\cdot\nabla\Psi)(\sigma_-)' = 0. \tag{27}$$

Inserting (27) in (25), there follows that

$$\nabla\cdot\mathsf{T} = 2\nabla\bar{p} - \nabla p_\parallel + \sigma_-\nabla\left(\frac{|B|^2}{2\mu_0}\right) + \sigma_-\mathbf{J}\times\mathbf{B}, \tag{28}$$

where we used Ampère's law (8). Substituting (28) into the equilibrium condition (7) gives

$$\nabla\bar{p} - \frac{|B|^2}{2\mu_0}\nabla\sigma_- = (1-\sigma_-)\mathbf{J}\times\mathbf{B}. \tag{29}$$

With the help of Eqs. (17) and (22), we obtain

$$(1-\sigma_-)\left[\Delta^*\Psi - \mu_0 I\mathcal{D} + \frac{\mu_0^2}{2}(I^2)'\right] = \frac{1}{2}\left[|\nabla\Psi|^2 + \mu_0^2 I^2\right](\sigma_-)' - \mu_0 g_{33}\bar{p}'. \tag{30}$$

After some algebra, we finally get an equilibrium equation for anisotropic plasmas with general symmetry with respect to x^3

$$\Delta^*\Psi - \mu_0 I\mathcal{D} - \frac{\sigma_-'}{2(1-\sigma_-)}|\nabla\Psi|^2 = -\frac{1}{2(1-\sigma_-)}\left[\mu_0^2 I^2(1-\sigma_-)\right]' - \frac{\mu_0 g_{33}\bar{p}'}{1-\sigma_-}, \tag{31}$$

which, in the isotropic limit ($p_\parallel = p_\perp = \bar{p} = p$), reduces to the Grad-Schlüter-Shafranov equation

$$\Delta^*\Psi - \mu_0 I\mathcal{D} = -\frac{\mu_0^2}{2}(I^2)' - \mu_0 g_{33}p'. \tag{32}$$

IV. THE CLEMENTE TRANSFORMATION

In the case $\sigma_- \neq 0$, Clemente has devised an ingenious method to solve the anisotropic equilibrium equation once a solution of the corresponding isotropic equilibrium equation is known.[11] Let us define the following auxiliary quantity:

$$U(\Psi) = \int_0^\Psi d\Phi\sqrt{1-\sigma_-(\Phi)}, \tag{33}$$

so that the chain rule furnishes

$$\frac{d\bar{p}}{dU} = \frac{\bar{p}'}{\sqrt{1-\sigma_-}} \tag{34}$$

and

$$\frac{d}{dU}\left[\frac{(1-\sigma_-)I^2}{2}\right] = \frac{\left[I^2(1-\sigma_-)\right]'}{2\sqrt{1-\sigma_-}}. \tag{35}$$

Multiplying Eq. (31) by $\sqrt{1-\sigma_-}$, and using (34) and (35), yields

$$\sqrt{1-\sigma_-}\Delta^*\Psi - \mu_0 I\mathcal{D}\sqrt{1-\sigma_-} - \frac{\sigma_-'\sqrt{1-\sigma_-}}{2(1-\sigma_-)}|\nabla\Psi|^2 = -\mu_0^2\frac{d}{dU}\left[\frac{(1-\sigma_-)I^2}{2}\right] - \mu_0 g_{33}\frac{d\bar{p}}{dU}. \tag{36}$$

According to definition (19) for the generalized Shafranov operator, a straightforward calculation results in

$$\sqrt{1-\sigma_-}\Delta^*\Psi = \Delta^* U + \frac{\sigma_-'}{2(1-\sigma_-)^{3/2}}|\nabla U|^2. \tag{37}$$

Similarly, the Laplacian in curvilinear coordinates gives

$$\frac{\sigma'_-\sqrt{1-\sigma_-}}{2(1-\sigma_-)}|\nabla\Psi|^2 = \frac{1}{2}\sigma'_-(1-\sigma_-)^{-3/2}|\nabla U|^2. \tag{38}$$

Substituting (37) and (38) into (36), we obtain an equilibrium equation in terms of the auxiliary function U

$$\Delta^* U - \mu_0 \mathcal{I}\mathcal{D} = -\frac{1}{2}\mu_0^2\frac{d\mathcal{I}^2}{dU} - \mu_0 g_{33}\frac{d\bar{p}}{dU}, \tag{39}$$

where we have defined a modified current function

$$\mathcal{I} = I\sqrt{1-\sigma_-}. \tag{40}$$

Note that Eq. (39) is formally equivalent to the Grad-Schlüter-Shafranov Eq. (32) provided $I \to \mathcal{I}$, $p \to \bar{p}$ and $\Psi \to U$. Hence, a known solution of the Grad-Schlüter-Shafranov equation can be used to obtain U. Once U is known, we can obtain σ_- by inverting the Clemente transformation (33). We shall limit ourselves to cases, where $|\sigma_-| < 1$. In order to invert (33), we have to make some assumption on the dependence of σ_- with Ψ. We can use a linear dependence as

$$\sigma_- = \sigma_0 + \sigma_1\Psi, \tag{41}$$

where σ_0 and σ_1 are known constants. Inserting (41) into (33) and integrating, we have

$$\Psi(U) = \frac{1}{\sigma_1}\left\{1-\sigma_0-\left[(1-\sigma_0)^{3/2}-\frac{3}{2}\sigma_1 U\right]^{2/3}\right\}. \tag{42}$$

Equation (39), just like the Grad–Schlüter–Shafranov equation, needs the previous specification of the functions $\bar{p}(U)$ and $\mathcal{I}(U)$ as well as adequate boundary conditions. Once we know a solution of Eq. (39) in the form $U(x^1, x^2)$, we use (41) and (42) so as to obtain the corresponding current function

$$I^2 = \frac{\mathcal{I}^2(U(\Psi))}{1-\sigma_0-\sigma_1\Psi}, \tag{43}$$

which is necessary to compute $|B|^2$ from Eq. (17) and the known expression for Ψ. Then we use $|B|^2$ to obtain σ_- from (24). We thus obtain a linear system

$$p_\parallel + p_\perp = 2p(\Psi), \tag{44}$$

$$p_\parallel - p_\perp = \frac{|B|^2}{\mu_0}(\sigma_0+\sigma_1\Psi), \tag{45}$$

whose solution gives us the pressures along directions parallel and perpendicular to the magnetic field.

V. ANISOTROPIC EQUILIBRIUM EQ. IN CYLINDRICAL COORDINATES

In the cylindrical coordinate system ($x^1 = R$, $x^2 = Z$, $x^3 = \phi$), the ignorable coordinate is $0 \le \phi < 2\pi$, such that the transversal flux function depends only on R and Z. Since $g = g_{33} = R^2$, the magnetic field and current density given, respectively, by (12), and (13), are

$$\mathbf{B}(R,Z) = \frac{1}{R}\,\hat{\mathbf{e}}_\phi \times \nabla\Psi - \frac{\mu_0 I}{R}\,\hat{\mathbf{e}}_\phi, \tag{46}$$

$$\mathbf{J}(R,Z) = \frac{1}{R}\,\hat{\mathbf{e}}_\phi \times \nabla I - \frac{1}{\mu_0 R}\Delta^*\Psi\,\hat{\mathbf{e}}_\phi, \tag{47}$$

where $\hat{\mathbf{e}}_\phi$ is the orthonormal basis vector and the Shafranov operator (19) is

$$\Delta^* = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial Z^2}, \tag{48}$$

with $\mathcal{D} = 0$ since the system is orthogonal.

Substituting (48) into (39), we have the equilibrium equation for the auxiliary function $U(R, Z)$

$$\frac{\partial^2 U}{\partial R^2} - \frac{1}{R}\frac{\partial U}{\partial R} + \frac{\partial^2 U}{\partial Z^2} = -\frac{1}{2}\mu_0^2\frac{d\mathcal{I}^2}{dU} - \mu_0 R^2\frac{d\bar{p}}{dU} = 0. \tag{49}$$

Solving this equation needs profiles for both $\mathcal{I}^2$ and $\bar{p}$ as functions of U. Let us consider linear profiles for them, in the form

$$\bar{p}(U) = p_0 - \frac{aB_0}{\mu_0 R_0^2}U, \tag{50}$$

$$\mathcal{I}^2(U) = \mathcal{I}_0^2 - \frac{2bB_0}{\mu_0^2}U, \tag{51}$$

where $\mathcal{I}_0$ and p_0 are constants, B_0 and R_0 is a characteristic magnetic field intensity and length, respectively, and a and b are real parameters.

The equilibrium Eq. (49) for these profiles reads

$$\frac{\partial^2 U}{\partial R^2} - \frac{1}{R}\frac{\partial U}{\partial R} + \frac{\partial^2 U}{\partial Z^2} = B_0\left(a\frac{R^2}{R_0^2} + b\right). \tag{52}$$

In the context of the Grad-Schluter-Shafranov equation, the solution of (52) was given by Solovev,[19] defining the dimensionless variables $r = R/R_0$ and $z = Z/R_0$

$$U(R,Z) = B_0 R_0^2\left[\frac{1}{2}(b+r^2)z^2 + \frac{1}{8}(a-1)(r^2-1)^2\right]. \tag{53}$$

We obtain the solution of the corresponding anisotropic equation by inverting the Clemente transformation. Adopting the linear relation (41) between σ_- and Ψ, we obtain from (42)

$$\Psi(U) = \frac{1}{\sigma_1}\left\{1-\sigma_0-\left[(1-\sigma_0)^{3/2}-\frac{3}{2}\sigma_1 B_0 R_0^2 \times\left(\frac{1}{2}(b+r^2)z^2+\frac{1}{8}(a-1)(r^2-1)^2\right)\right]^{2/3}\right\}. \tag{54}$$

We simplify matters by choosing $\sigma_0 = 0$ and $\sigma_1 = 1/(B_0 R_0^2)$, so that the normalized flux function $\psi = \Psi\sigma_1$ is

$$\psi(r,z) = 1 - \left[1 - \frac{3}{2}\left(\frac{1}{2}(b+r^2)z^2 + \frac{1}{8}(a-1)(r^2-1)^2\right)\right]^{2/3}. \tag{55}$$

The magnetic axis is located in the point where the function flux is extremal, this is, at the point $R = R_0$ and $Z = 0$. From Eq. (16), we can calculate the components of the magnetic field, we obtain

$$B_R = -\frac{B_0 z}{r}\frac{(b+r^2)}{\left[1-\frac{3}{4}(b+r^2)z^2-\frac{3}{16}(a-1)(r^2-1)^2\right]^{1/3}}, \tag{56}$$

$$B_Z = B_0\frac{z^2+\left[(a-1)(r^2-1)\right]/2}{\left[1-\frac{3}{4}(b+r^2)z^2-\frac{3}{16}(a-1)(r^2-1)^2\right]^{1/3}}, \tag{57}$$

$$B_\phi = \frac{B_0}{r}\frac{\sqrt{1-b\left[(b+r^2)z^2+(a-1)(r^2-1)^2/4\right]}}{\left[1-\frac{3}{4}(b+r^2)z^2-\frac{3}{16}(a-1)(r^2-1)^2\right]^{1/3}}, \tag{58}$$

where we use $B_0 = -\mu_0\mathcal{I}_0/R_0$. We can solve the linear system (44) and (45) for the pressures, yielding

$$p_\parallel = \frac{2\mu_0 p + |B|^2\sigma_1\Psi}{2\mu_0}, \tag{59}$$

$$p_\perp = \frac{2\mu_0 p - |B|^2\sigma_1\Psi}{2\mu_0}. \tag{60}$$

Finally, we can calculate the pressures along and across the magnetic field. For the case $\mathcal{I} = 0$ in the equatorial plane ($Z=0$), the pressure yields

$$\frac{p_{\parallel,\perp}}{p_0} = 1-\beta^{-1}a\frac{(a-1)}{4}(r^2-1)^2 \pm \frac{\beta^{-1}(a-1)^2(r^2-1)^2}{4\left[1-3(a-1)(r^2-1)^2/16\right]^{2/3}} \times\left[1-\left(1-3\frac{(a-1)}{16}(r^2-1)^2\right)^{2/3}\right], \tag{61}$$

where $p_\parallel$ ($p_\perp$) corresponds to the upper (lower) sign, and the plasma beta is

$$\beta^{-1} = \frac{B_0^2/2\mu_0}{p_0}. \tag{62}$$

In Fig. 2, we have the curves of pressure for the equatorial plane for different values of the parameter a and $\beta = 2$. The pressure parallel to the magnetic field is greater that the perpendicular pressure for all cases. We can interpret this as a result of the preferential direction of motion created by the magnetic field. In the center of the cylinder, we have the largest difference between the pressures and both go to the same maximum value at the magnetic axis. The parameter a is greater than one, otherwise the pressure profile $\bar{p}$ would have negative value, having no physical meaning.

The parameter a has a set of possible values, which we choose so that: (i) none of the pressures have negative values and (ii) the solution represents closed magnetic surfaces. We found that the set of values that respect these conditions is $1.0 < a \leq 3.0$. The difference of the parallel and perpendicular pressures is greater in the geometric axis of the cylinder ($r=0$). In Fig. 3, we show how the difference of the pressures increases with a, for the point $r=0.0001$, what can also be seen by comparing Figs. 2(a) and 2(c).

VI. SPHERICAL COORDINATES

In this section, we consider spherical coordinates ($x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$), with $0 \leq \varphi < 2\pi$ as the ignorable coordinate and, since this system is orthogonal, $\mathcal{D} = 0$. The metric tensor is such that $g_{33} = r^2\sin^2\theta$ and $g = r^4\sin^2\theta$. Hence, the equilibrium Eq. (39) reads

$$\frac{\partial^2 U}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\theta}\right) = -\frac{\mu_0^2}{2}\frac{d\mathcal{I}^2}{dU}-\mu_0 r^2\sin^2\theta\frac{d\bar{p}}{dU}. \tag{63}$$

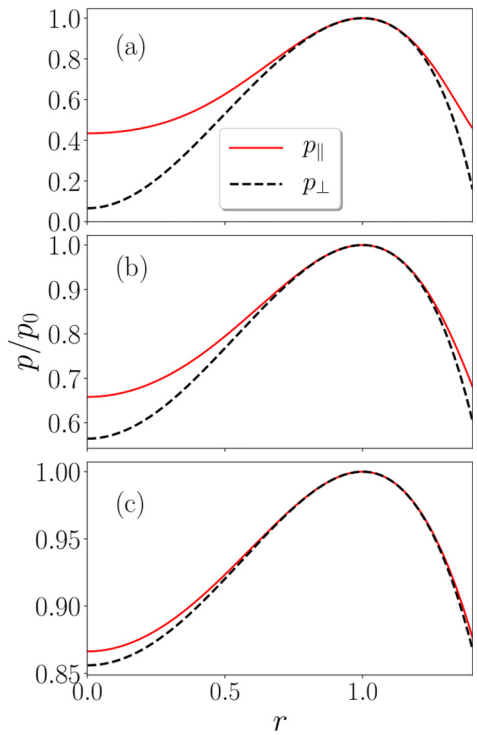

FIG. 2. Radial profiles of normalized parallel (full lines) and perpendicular (dash lines) pressures for $\beta = 2$ and (a) $a = 3$, (b) $a = 7/3$, and (c) $a = 5/3$.

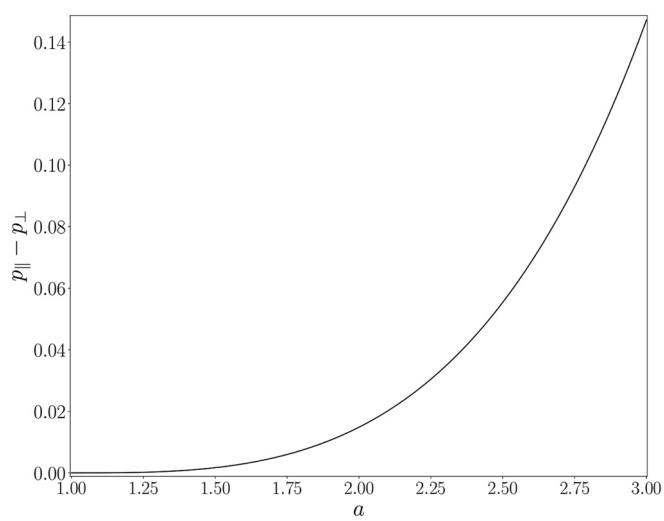

FIG. 3. Difference of pressures as a function of the parameter a for the cylindrical solution.

We consider essentially the same profiles proposed by Morikawa[21] in his study of a spherical plasma ($0 \le r \le a$) surrounded by a force-free medium ($a \le r < 1$) and a spherical conducting shell at $r = 1$. In this section, we shall consider that the radial distances are normalized by the radius of the conducting shell. For the plasma region

$$\bar{p}(U) = p_0 + \frac{\kappa^2}{\mu_0} U, \tag{64}$$

$$\mathcal{I}^2(U) = \mathcal{I}_0^2 + \frac{\lambda^2}{\mu_0^2} U^2, \tag{65}$$

and, for the force-free medium

$$\bar{p}(U) = 0, \tag{66}$$

$$\mathcal{I}^2(U) = \mathcal{I}_0^2 + \frac{\lambda^2}{\mu_0^2} U^2, \tag{67}$$

where κ and λ are both positive constants, as p_0 and $\mathcal{I}_0$. In the force-free medium, we have $p_\perp = p_\parallel = 0$, such that $\bar{p}$ vanishes identically.

The equilibrium Eq. (63) for the plasma region is

$$\frac{\partial^2 U}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\theta}\right) + \lambda^2 U = -\kappa^2 r^2 \sin^2\theta, \tag{68}$$

and for the force-free medium, the corresponding equation is obtained by making $\kappa = 0$ in (68), that is, the homogeneous equation for (68). The solution given by Morikawa[21] can be written as

$$U_1(r,\theta) = \left[A\left(\frac{\sin\lambda r}{\lambda r} - \cos\lambda r\right) - \frac{\kappa^2 r^2}{\lambda^2}\right]\sin^2\theta, \tag{69}$$

$$U_2(r,\theta) = \left[C\left(\frac{\sin\lambda r}{\lambda r} - \cos\lambda r\right) + D\left(\frac{\cos\lambda r}{\lambda r} + \sin\lambda r\right)\right]\sin^2\theta, \tag{70}$$

where U_1 is the solution for the plasma medium, U_2 for the force-free medium, and A, C, and D are the integration constants.

The pressure must vanish at the plasma boundary ($r = a$), so that (73) gives

$$A = \frac{\kappa^2 a^2/\lambda^2}{(\sin\lambda a/\lambda a) - \cos\lambda a}. \tag{71}$$

Another boundary condition is that $U_2 = 0$ at the spherical conducting shell at $r = 1$, which leads to the transcendental equation $\tan\lambda = \lambda$, whose first positive roots are denoted $\lambda_1 \approx 4.493$ and $\lambda_2 \approx 7.725$. We choose $\lambda = \lambda_2$ so as to have a single magnetic axis inside the plasma region $0 < r < a$. The boundary condition $U_2(r = a,\ \theta) = 0$ for $C \neq 0$ and $D = 0$ leads to a similar equation, namely, $\tan\lambda a = \lambda a$, which fixes the plasma radius as $a = \lambda_1/\lambda_2 = 0.5816$.[21]

Moreover, λ may assume any value in the interval $0 \le \lambda \le \lambda_2$. In the isotropic case, Morikawa showed that the limit $\lambda \to 0$ is a solution known in hydrodynamics as the Hill vortex model,[21] which is a useful model for field-reversed configurations and spheromak equilibria.[22] The limit $\lambda \to \lambda_2$ gives us a force-free medium. Finally imposing that $\partial U/\partial r$ is continuous across the plasma boundary gives the remaining integration constant

$$C = \frac{\kappa^2 a^2}{\lambda^2 \lambda_1 \sin\lambda_1}\left[\frac{\lambda a \sin\lambda a}{(\sin\lambda a/\lambda a) - \cos\lambda a} - 3\right], \tag{72}$$

which also fixes the value of κ as a function of λ. The final solution reads

$$U_1(r,\theta) = C\lambda_1 \sin\lambda_1 \times \frac{a^2[(\sin\lambda r/\lambda r) - \cos\lambda r] - r^2[(\sin\lambda a/\lambda a) - \cos\lambda a]}{\lambda a \sin\lambda a - 3[(\sin\lambda a/\lambda a) - \cos\lambda a]}\sin^2\theta, \tag{73}$$

$$U_2(r,\theta) = C\left(\frac{\sin\lambda_2 r}{\lambda_2 r} - \cos\lambda_2 r\right)\sin^2\theta. \tag{74}$$

Making use of the Clemente transformation for the linear dependence of σ_- and Ψ given by (42), we obtain the anisotropic solutions for the plasma and force-free regions, with $\sigma_0 = 0$, given, respectively, by

$$\Psi_1(r,\theta) = \frac{1}{\sigma_1}\left\{1 - \left[1 - \frac{3}{2}\sigma_1 C \lambda_1 \sin\lambda_1 \times \frac{a^2[(\sin\lambda r/\lambda r) - \cos\lambda r] - r^2[(\sin\lambda a/\lambda a) - \cos\lambda a]}{\lambda a \sin\lambda a - 3[(\sin\lambda a/\lambda a) - \cos\lambda a]} \times \sin^2\theta\right]^{2/3}\right\}. \tag{75}$$

$$\Psi_2(r,\theta) = \frac{1}{\sigma_1}\left\{1 - \left[1 - \frac{3}{2}\sigma_1 C\left(\frac{\sin\lambda_2 r}{\lambda_2 r} - \cos\lambda_2 r\right)\sin^2\theta\right]^{2/3}\right\}. \tag{76}$$

In Fig. 4, we show for the plasma region, the magnetic surfaces in the isotropic and anisotropic cases, when $\lambda = \lambda_2$ (the solution being similar in both plasma and force-free regions). The flux function in

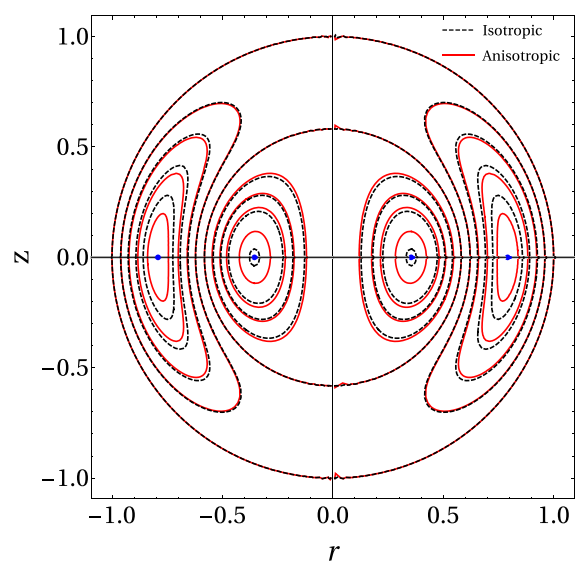

FIG. 4. Flux surfaces for the isotropic (dashed black lines) and anisotropic (full red lines) solutions with $\lambda = \lambda_2$. We indicate the positions of the magnetic axes by the blue dots.

the plasma region has a maximum at the magnetic axis $r = r_0$, given by solving the transcendental equation

$$\frac{a^2}{r_0}\left(\cos\lambda r_0 - \frac{\sin\lambda r_0}{\lambda r_0} + \lambda r_0 \sin\lambda r_0\right) = 2r_0\left(\frac{\sin\lambda a}{\lambda a} - \cos\lambda a\right). \tag{77}$$

Making use of the same procedure as in Sec. V, we calculate the parallel and perpendicular pressures corresponding to the plasma region, our results being shown in Fig. 5 for the equatorial plane $\theta = \pi/2$. In this calculation, we defined $p_0 = C^2/\mu_0$ and choose $C = 1/\sigma_1$ and $\mathcal{I}_0\mu_0 = C$. In the same way as in the cylindrical case discussed in the Sec. V, $p_\parallel$ is larger than $p_\perp$. Moreover the difference between $p_\parallel$ and $p_\perp$ increases as λ approaches λ_2, i.e., when we move towards the force-free case.

From Eq. (16), we obtained the components of the magnetic field, such that its radial component is always zero in the equatorial plane $\theta = \pi/2$ and the radial profiles of the poloidal (θ) and toroidal (ϕ) components are shown in Figs. 6 and 7, respectively, and compared with the isotropic case. We observed that for small radii, the poloidal component in the anisotropic case is slightly larger than for the isotropic case, but after the magnetic axis position, the anisotropic result is slightly smaller. Moreover, we observe that at the magnetic axis radius there is also a poloidal field reversal. The difference between iso and anisotropic values increases very little with λ.

On the other hand, for the toroidal field component, the anisotropic result is always lower than that for the isotropic one. The difference is most pronounced at $r \approx 0.3$ and decreases as we approach $r = 0$ and $r = a$. There are no field reversals in both cases.

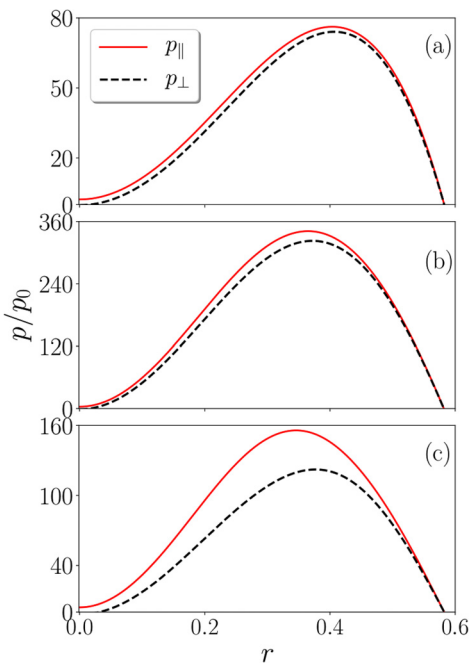

FIG. 5. Radial profiles of the parallel (full lines) and perpendicular (dashed lines) normalized pressures for (a) $\lambda = 3.0$, (b) $\lambda = 7.0$, and (c) $\lambda = 7.5$.

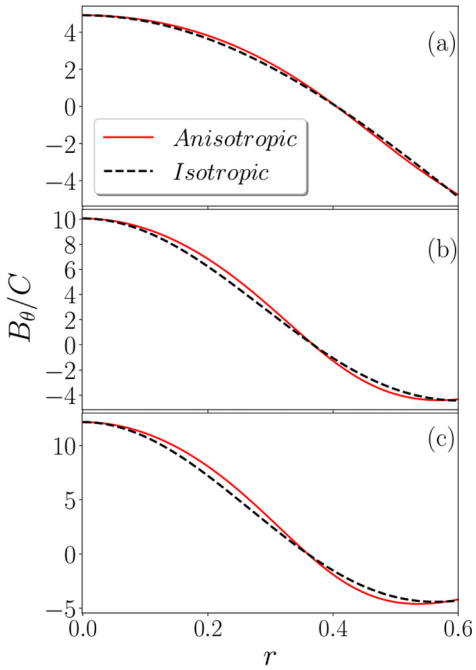

FIG. 6. Radial profiles of the poloidal magnetic field component for the anisotropic (full lines) and isotropic cases (dashed lines), where (a) $\lambda = 3.0$, (b) $\lambda = 7.0$, and (c) $\lambda = 7.5$.

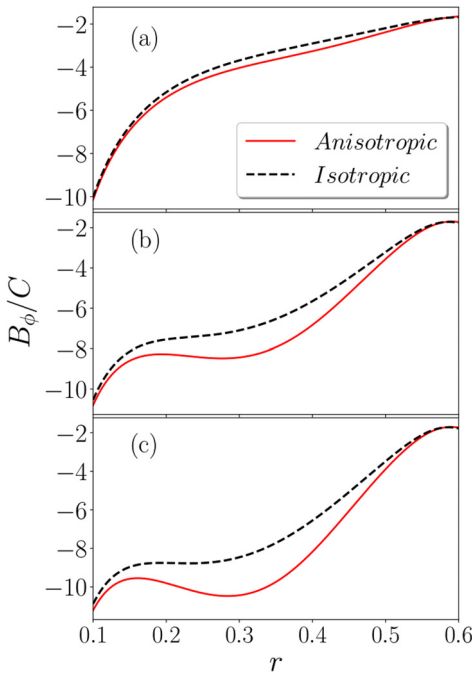

FIG. 7. Radial profiles of the toroidal magnetic field component for the anisotropic (full lines) and isotropic cases (dashed lines), where (a) $\lambda = 3.0$, (b) $\lambda = 7.0$, and (c) $\lambda = 7.5$.

The previous results refer to the plasma region. In the force-free medium, the components of the field are given by

$$B_{r,2}(r,\theta) = -\frac{2C\cos\theta}{r^2}\frac{\sin\lambda_2 r/\lambda_2 r - \cos\lambda_2 r}{\left[1-\frac{3}{2}\sin^2\theta\left(\frac{\sin\lambda_2 r}{\lambda_2 r}-\cos\lambda_2 r\right)\right]^{1/3}}, \tag{78}$$

$$B_{\theta,2}(r,\theta) = \frac{C\sin\theta}{r^2}\frac{\cos\lambda_2 r - \sin\lambda_2 r/\lambda_2 r + \lambda_2 r\sin\lambda_2 r}{\left[1-\frac{3}{2}\sin^2\theta\left(\frac{\sin\lambda_2 r}{\lambda_2 r}-\cos\lambda_2 r\right)\right]^{1/3}}, \tag{79}$$

$$B_{\phi,2}(r,\theta) = \frac{-C}{r\sin\theta}\frac{\sqrt{1+\lambda^2\sin^4\theta(\sin\lambda_2 r/\lambda_2 r - \cos\lambda_2 r)^2}}{\left[1-\frac{3}{2}\sin^2\theta\left(\frac{\sin\lambda_2 r}{\lambda_2 r}-\cos\lambda_2 r\right)\right]^{1/3}}. \tag{80}$$

VII. CONCLUSIONS

In this paper, we further developed a technique to study MHD equilibria of symmetric systems with anisotropic pressure, to orthogonal as well as nonorthogonal coordinate systems. It is possible to obtain a solution of the anisotropic case from a known solution of the equilibrium equation in the isotropic case using a transformation due to Clemente, viz., Eq. (33).

We applied this technique to the case of a cylindrical plasma with linear profiles for pressure and current. We obtained that the parallel pressure is greater than the perpendicular pressure in all radial positions except at the magnetic axis, where this difference vanishes. Moreover, the position of the magnetic axis itself does not change for the iso and anisotropic cases.

Second, we consider the case of spherical coordinates, where we suppose a linear profile for the pressure and a quadratic profile for current. In the isotropic case, the solution has a tunable parameter whose variation enables us to go from a Hill vortex solution to a force-free medium. The parallel pressure is also larger than the perpendicular pressure in the plasma region, and they are no longer equal at the magnetic axis. The difference of parallel and perpendicular pressures increases as we change parameters so as to move towards a force-free medium.

In the spherical case, the poloidal magnetic field shows a reversal of sign at the position of the magnetic axis. Moreover, when compared to the poloidal component of the magnetic field of the isotropic case, the intensity is slightly larger in the anisotropic solution before the magnetic axis position and slightly smaller after the magnetic axis. The toroidal magnetic field does not show sign reversal, and the difference between the iso and anisotropic solutions is relatively small near the center and the plasma boundary, being more appreciable around the magnetic axis. This difference also increases as we change parameters so as to approach the force-free case.

In summary, the method we described in this paper can be applied to any coordinate system for which the equilibrium equation has a closed-form, analytical solution. Even though there are few such solutions, they can be nevertheless be used for benchmarking fully numerical solutions of the anisotropic equilibrium equation.

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