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Citation: Physics of Plasmas 23, 122503 (2016); doi: 10.1063/1.4971218

View online: http://dx.doi.org/10.1063/1.4971218

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Stationary MHD equilibria describing azimuthal rotations in symmetric plasmas

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(Received 25 August 2016; accepted 15 November 2016; published online 5 December 2016)

We consider the stationary magnetohydrodynamical (MHD) equilibrium equation for an axisymmetric plasma undergoing azimuthal rotations. The case of cylindrical symmetry is treated, and we present two semi-analytical solutions for the stationary MHD equilibrium equations, from which a number of physical properties of the magnetically confined plasma are derived. Published by AIP Publishing. [http://dx.doi.org/10.1063/1.4971218]

I. INTRODUCTION

Stationary magnetohydrodynamical (MHD) equilibria are characterized by time-independent fields but allowing a constant velocity. One example is azimuthal rotations in axisymmetric plasmas.1 In tokamaks, for example, the use of neutral beam injection for non-inductive driving purposes causes toroidal rotation in the confined plasma.2,3 Magnetic stars are other examples of confined plasmas under some kind of rotation.4 One important problem in astrophysics for stars are other examples of confined plasmas under some kind of rotation.5

While static MHD equilibria have been extensively investigated for over five or six decades by now, stationary equilibria have relatively fewer investigations, both theoretical and numerical. Maschke and Perrin6 have proposed a treatment for azimuthal plasma rotations and obtained an equilibrium equation in cylindrical coordinates which, in the static limit, reduces to the usual Grad-Schlüter-Shafranov equation for the poloidal magnetic flux.7,8 However, the magnetic (flux) surfaces no longer coincide with the isoobaric (constant pressure) surfaces in the rotating case.

The explicit presence of the plasma density in the stationary MHD equilibrium equation demands the use of additional thermodynamical hypotheses. Maschke and Perrin have considered that either the entropy or the temperature can be taken as surface quantities. In both cases Maschke and Perrin have obtained analytical solutions by assuming linear profiles for both the modified pressure and the current flux.5

One can hardly overestimate the usefulness of analytical or semi-analytical (i.e., explicit separation of variables, with one of them satisfying a two-point boundary value problem with numerical solution) solutions for the MHD equilibrium equations. Since in most practical solutions a full numerical solution of a nonlinear complicated boundary value problem is necessary, analytical solutions can benchmark computational schemes of solution allowing for important convergence, stability and accuracy tests.9,10

In the case of temperature as a surface quantity, Clemente and Farengo have obtained a semi-analytical solution in the case of quadratic profiles for the modified pressure and the current flux.11 Another analytical solution in power series has been obtained by Pantuso and Sudano.12 Starting from a general form of this stationary equilibrium equation in curvilinear coordinates,13 Viana, Clemente, and Lopes have obtained an approximate analytical solution for the corresponding equation in spherical coordinates.14

The case of entropy as a surface quantity has been less studied,15 since there are no further solutions, besides that obtained by Maschke and Perrin themselves. In the present work we present two semi-analytical solutions for the stationary equilibrium equation with a quadratic profile for the current flux function and linear and quadratic profiles for the modified pressure.

This paper is organized as follows: in Section II we outline the basic equations necessary to derive the stationary equilibrium equation and introduce the surface quantities which follow from assuming axisymmetry. Sections III and IV show the equilibrium equation in cylindrical coordinates as well as the two semi-analytical solutions obtained for quadratic and linear profiles, respectively. Sec. V is devoted to our conclusions.

II. EQUILIBRIUM STATIONARY MHD EQUATION

We start from the ideal stationary MHD equilibrium equations (SI units are used)9,10,16

\[ \nabla \cdot (\rho v) = 0, \]  
\[ \rho (v \cdot \nabla) v = -\nabla p + J \times B, \]  
\[ \nabla \times E = 0, \]  
\[ \nabla \times B = \mu_0 J, \]  
\[ E + v \times B = 0, \]  
\[ p = \rho \beta T, \] \n
where \( \rho, p, J, E, \) and \( B \) represent the plasma density, scalar pressure, current density, electric field, and magnetic induction, respectively, the latter being chosen so as to satisfy the condition \( \nabla \cdot B = 0. \)
In the following discussion, we consider axisymmetric plasma configurations in cylindrical coordinates \((R, Z, \phi)\), i.e., we suppose an ignorable coordinate \(0 \leq \phi < 2\pi\), such that surface quantities do not depend on it.\(^{17,18}\) A representation for the magnetic induction in this case is

\[
B(R, Z) = \frac{1}{R} \varepsilon_{\phi} \times \nabla \Psi - \frac{\mu_0 d}{R} \varepsilon_{\phi},
\]

(7)

where the surface quantities are \(\Psi(R, Z)\) (poloidal flux function) and \(I(R, Z)\) (current flux function). From Ampère’s law (4) there follows a representation for the current density

\[
J(R, Z) = \frac{1}{R} \varepsilon_{\phi} \times \nabla I - \frac{1}{\mu_0 R} \Delta^* \Psi \varepsilon_{\phi},
\]

(8)

where

\[
\Delta^* \Psi = R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial Z^2},
\]

(9)

is the Shafranov elliptic operator.

We suppose from now on that the rotation is purely azimuthal, i.e., \(v = R \Omega \varepsilon_{\phi}\), a choice compatible with the mass continuity equation (1). Eliminating the electric field using generalized Ohm’s law (5) and substituting in Faraday’s law (3) it results that \(\nabla \Omega \times \nabla \Psi = 0\), such that \(\Omega = \Omega(\Psi)\) is also a surface quantity and the magnetic surfaces undergo rigid rotations, though with different angular velocities, a fact known as Ferraro’s isorotation law.\(^{19}\) As a result, we have that \(\nabla \Omega = \Omega \nabla \Psi\), where the primes stand, from now on, for derivatives with respect to \(\Psi\).

It is actually possible to have a more general rotation satisfying mass continuity equation (1), describing a composite rotation with azimuthal (toroidal) as well as transversal (poloidal) components.\(^{2,20}\) A general equilibrium equation with differentially varying radial electric fields was proposed by Tasso and Thomolopoulos, who showed the nonexistence of axisymmetric equilibria with purely poloidal flows or nonparallel flows with isotropic magnetic surfaces and omnigenous equilibria.\(^{21}\)

Dotting (2) with \(B\) results in \(B \cdot \nabla p = -\rho B \cdot (v \cdot \nabla) v\), which is nonzero for a finite velocity. Hence the isobaric (constant pressure) surfaces are no longer magnetic (constant flux) surfaces, as it happens in the static case. In other words, the plasma pressure is no longer a surface quantity for stationary equilibria. Instead of the pressure, we can use another thermodynamical variable to play this role. Two of such choices have been proposed by Maschke and Perrin, namely, the entropy per mass unit \(s\) or the temperature \(T\).\(^{6}\) We shall assume that the former, i.e., the specific entropy is a surface quantity: \(s = s(\Psi)\), such that \(\nabla s = s' \nabla \Psi\).

The stationary equilibrium equation follows from (2):

\[
\left[ \Delta^* \Psi - \frac{1}{\mu_0^2} (F')^\prime + \mu_0 R^4 \rho \Omega' - \mu_0 R^2 \rho T s' \right] \nabla \Psi
= -\mu_0 R^2 \rho \nabla \Theta,
\]

(10)

where we define the following surface quantity:

\[
\Theta = h - \frac{1}{2} R^2 \Omega^2,
\]

(11)

where \(h\) is the specific enthalpy, satisfying \(dh = T ds + (d\rho/\rho)\).

We eliminate the density in (10) from (6) and the thermodynamical relation \((\rho\varepsilon)\) is the specific internal energy

\[
T = \frac{\partial e}{\partial s} = \frac{\rho \varepsilon^{\prime\prime\prime\prime}}{\gamma - 1} \frac{dA}{ds}.
\]

(12)

\[
\Delta^* \Psi + \frac{1}{2} \mu_0^2 (F')^\prime + \mu_0 R^2 \left( 1 + \frac{R^2 \lambda^2}{2\Theta} \right)^\eta - \Theta \frac{\partial^2 \Psi}{\partial Z^2} = 0,
\]

(13)

where \(\eta = \frac{\gamma}{\gamma - 1}\) and

\[
\Pi = \left[ \left( \frac{\Theta}{\eta} \right) A^{1-\eta} \right]^{\prime}
+ R^2 \left[ \left( \frac{\Theta}{\eta} \right) A^{1-\eta} \frac{\Omega \Psi}{\Theta} + \left( \Theta \right) \left( A^{1-\eta} \right)^\prime \frac{\Theta^2}{\Theta^2} \right].
\]

(14)

The equilibrium equation (13) can only be solved if we specify \textit{a priori} profiles for the four surface quantities \(I(\Psi), \Omega(\Psi), \Theta(\Psi)\) and \(A(\Psi)\). In order to simplify this task Maschke and Perrin\(^{6}\) have considered the case for which

\[
\frac{\Omega^2}{\Theta} = \frac{\omega^2}{\ell^2},
\]

(15)

where \(\omega\) is a rotation parameter and \(\ell\) is a characteristic system length. Using (15) in (14) gives an equilibrium equation involving only two surface quantities

\[
\Delta^* \Psi + \frac{1}{2} \mu_0^2 (F')^\prime + \mu_0 R^2 \left( 1 + \frac{R^2 \lambda^2}{2\Theta} \right)^\eta = 0,
\]

(16)

where we have defined another surface quantity,

\[
G(\Psi) = \left( \frac{\Theta}{\eta} \right) A^{1-\eta}.
\]

(17)

The condition (15) imposes a limitation to the rotation velocities allowed by this description. The requirement that \(\Theta\) be positive results in

\[
h > \frac{1}{2} \Theta \ell^2 g_{33}.
\]

(18)

Considering the sound velocity

\[
c_s^2 = \frac{\partial p}{\partial \rho_s \mid_{s}} = \gamma A(s) \rho v^{\prime\prime\prime\prime} = h(\gamma - 1),
\]

(19)

the condition (18) can be rewritten as

\[
\mathcal{M}^2 < \frac{2}{\gamma - 1},
\]

(20)
where the Mach number is $\mathcal{M} = v/c_s$. Using $\gamma = 5/3$ there follows that $\mathcal{M} < \sqrt{3} \approx 1.73$. Moreover, the condition (15) allows us to express the parameter $\omega$ as

$$\omega^2 = \frac{\ell^2}{R^4} \left[ \frac{(\gamma - 1)\mathcal{M}^2}{1 - (\gamma - 1)\mathcal{M}^2/2} \right].$$

(21)

### III. SOLUTION FOR A QUADRATIC PRESSURE

Maschke and Perrin have chosen linear profiles for both $G$ and $\tilde{f}$, obtaining an exact solution for eq. (16). In this paper, we solve this equation for two different sets of profiles. The first set has quadratic profiles for the centrifugally corrected pressure and quadratic profiles for the squared current function:

$$G(\Psi) = \frac{P}{2\rho_0 R_0^2} \Psi^2, \quad \tilde{f}(\Psi) = \frac{M}{\rho_0 R_0^2} \Psi^2,$$

(22)

where $P$ and $M$ are constants. While $P$ is always positive, if $M$ is positive (negative) we are dealing with paramagnetic (diamagnetic) plasmas, $M = 0$ corresponding to a vacuum field. Substituting these profiles in (16) we have

$$\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = -M\Psi - P r^2 \left(1 + \frac{\omega^2 r^2}{2}\right) \Psi,$$

(23)

where we define nondimensional variables $r = R/R_0$ and $z = Z/L$, $L$ being a characteristic length.

We separate variables as $\Psi(r, z) = f(r)g(z)$, in such a way that the functions $f$ and $g$ obey the following ordinary differential equations:

$$\frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \left[ M - \lambda^2 + P r^2 \left(1 + \frac{\omega^2 r^2}{2}\right) \right] f = 0,$$

(24)

$$\frac{\partial^2 g}{\partial z^2} + \lambda^2 g = 0,$$

(25)

where $\lambda^2$ is a separation constant.

Let us consider the following boundary condition: a cylindrical conducting drum of radius $r = 1$ and height $z = 1$, for which the eigenfunctions and eigenvalues for the $z$-dependent part are $g_n(z) = \sin(n\pi z)$ and $\lambda^2 = n^2\pi^2$, respectively, where $n$ is a positive integer. Physically, the multiple eigensolutions correspond to the possible existence of various magnetic axes. For the sake of simplicity we will assume the existence of just one such axis, and hence, we select $n = 1$ hereafter.

With such eigenvalue Eq. (24) reads

$$\frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \left[ M - \pi^2 + P r^2 \left(1 + \frac{\omega^2 r^2}{2}\right) \right] f = 0,$$

(26)

which is a two-point boundary value problem which has been numerically solved for $f(0) = f(1) = 0$. Once we fix the value of parameter $M$, say, at $M = 1$, this problem has solution for selected values of the parameter $P$, which are the eigenvalues corresponding to the radial equation. Assuming again a single magnetic axis we compute only the lowest eigenvalue for $P$, which is plotted in Fig. 1(a) as a function of the plasma angular velocity for different values of $M$.

The solution (analytical in $z$ and numerical in $r$) we obtained for $\Psi(r, z)$ allows us to draw a radial profile of the centrifugally corrected pressure $G$, given by Eq. (17), at the midplane $z = 1/2$, the result being plotted in Fig. 2(a) for different values of the angular velocity $\omega$. The profile for $G$ vanishes at both $r = 0$ and $r = 1$ and has a maximum $r = r_a$ corresponding to the magnetic axis position. As the angular velocity increases this position changes outwards, corresponding to an overall centrifugal displacement of the magnetic surfaces.

In Fig. 3, we plot the normalized outward displacement of the magnetic axis as a function of the angular velocity for different values of the parameter $M$, indicating that, in fact, the magnetic axis position changes monotonically with $\omega$. Hence, even though our theoretical description holds only for Mach numbers less than $\sqrt{3}$, it appears that the same behavior would be observed for higher values of $\omega$, provided

![FIG. 1. Allowed values of the pressure parameter $P$ as a function of the plasma angular velocity $\omega$ for different values of the parameter $M$. (a) is for a quadratic pressure profile and (b) is for a linear pressure profile.](image1.png)

![FIG. 2. Radial profiles of the centrifugally corrected pressure $G$ at the midplane $z = 1/2$ for different values of the angular velocity. (a) is for a quadratic pressure profile and (b) is for a linear pressure profile. $G_0$ is the value of $G$ at the magnetic axis.](image2.png)
the velocities are much less than \( c \), as our non-relativistic description remains valid. Moreover, we see that the radial displacement is comparatively shorter for \( M = 1 \) (paramagnetic case) than for \( M = 0 \) (vacuum field), although this difference shrinks for both small and large values of \( \omega \).

The magnetic field components are given by (7), in terms of the flux function, as

\[
\mathbf{B}(R, Z) = \left( -\frac{1}{R} \frac{\partial}{\partial Z} \frac{1}{R} \frac{\partial}{\partial R}, M \psi \right).
\]  

(27)

The radial profile for the normalized values of the vertical field in the midplane \( z = 1/2 \) are shown in Fig. 4(a) for different values of the angular velocity. In both static and rotating cases there is a field reversal whose radial location also shifts outwards due to the centrifugal effect. From (27) the profile for the azimuthal component is essentially the same as for the centrifugally corrected pressure.

The current density components are given by (8) as

\[
\mathbf{J}(R, Z) = \left( -\frac{1}{R} \frac{\partial}{\partial Z} \frac{1}{R} \frac{\partial}{\partial R}, \Delta \psi \right).
\]  

(28)

The normalized values of vertical current density in the midplane \( z = 1/2 \) for different values of the angular velocity and \( M = 1 \). (a) is for a quadratic pressure profile and (b) is for a linear pressure profile. \( J_0 \) is the vertical current density at \( r = 0 \).

IV. SOLUTION FOR A LINEAR PRESSURE

The second set we have considered in this paper has a linear profile for the centrifugally corrected pressure and a quadratic profile for the squared current function:

\[
G(\Psi) = \frac{P}{2 \mu_0 R_0^2} \psi^2, \quad I^2(\Psi) = \frac{M}{\mu_0 R_0^2} \psi^2,
\]  

(29)

with which Eq. (16) becomes

\[
\frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R} \frac{\partial \Psi}{\partial R} + \frac{\partial^2 \Psi}{\partial Z^2} = \frac{M}{R_0^2} \Psi - \frac{PR^2}{R_0^2} \left( 1 + \frac{\omega^2 R^2}{2R_0^2} \right)^\eta.
\]  

(30)

A first separation of variables is \( \Psi(r, z) = h(r) + g(r) \), where the variables are also normalized, and the functions \( h \) and \( g \) obey the following equations:

\[
\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} + \frac{R_0^2 \partial^2 h}{L^2 \partial z^2} + M h = 0,
\]  

(31)

\[
\frac{d^2 g}{dr^2} - \frac{1}{r} \frac{dg}{dr} + M g - P r^2 \left( 1 + \frac{1}{2} \frac{\omega^2 r^2}{L^2} \right)^\eta = 0.
\]  

(32)
In Eq. (31) we use the following ansatz \( h(r, z) = R(r)Z(z) \), yielding the following ordinary differential equations:

\[
\frac{d^2Z}{dz^2} + \frac{L^2 M}{2R_0^2} Z = 0, \tag{33}
\]

\[
\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \frac{M}{2} R = 0. \tag{34}
\]

We will consider the same boundary conditions as we did in the previous case. The eigenfunctions for the \( z \)-part of the solution are \( Z_n(z) = \sin(n\pi z) \), where \( n \) is a positive integer. Moreover (34) has an eigensolution regular at the origin given by \( R_m(r) = r J_1(\gamma_{1m} r) \), where \( \gamma_{1m} \) is the \( m \)-th root of the Bessel function \( J_1 \), with \( m = 1, 2, \ldots \). As in the previous case, we assume the existence of just one magnetic axis, and hence, we select \( n = m = 1 \).

Equation (32), on the other hand, has no analytical solution, and we solve it numerically as a two-point boundary value. Like in the previous case, given the value of the parameter \( M \), (32) has solutions compatible with the boundary conditions only for selected values of \( P \), which we plot in Fig. 1(b) as a function of the angular velocity.

In Fig. 2(b) we show the radial profile of the centrifugally corrected pressure at the midplane \( Z = L/2 \) for different values of the \( \omega \), also showing an outward displacement of the magnetic axis. The normalized magnetic axis displacement is plotted in Fig. 3(b) as a function of \( \omega \) for different values of \( M \).

The radial profile for the \( Z \)-components of the magnetic field in the midplane is shown in Fig. 4(b) as a function of \( \omega \). There is a field reversal radius that increases with rotation. This can also be seen in the corresponding profile for the plasma density component [Fig. 5(b)].

A different behavior is observed, though, for the \( \phi \)-component of the plasma density [Fig. 6(b)]. While in the static case there is no current reversal, it appears for a rotating plasma with \( \omega = 1 \). Current reversal has been described in many static configurations, and in our case, it has appeared due to the toroidal rotation, nevertheless, linked with the particular form of the profiles we used here. It has not been seen in other cases. Finally, the shapes of the magnetic flux surfaces [cf. Fig. 7(b)] are similar to the previously considered profile.

V. CONCLUSIONS

In this work, we obtained two new solutions to the Maschke-Perrin equation describing stationary MHD equilibria of azimuthally rotating plasmas in cylindrical coordinates. These solutions correspond to two different profiles for the modified pressure, namely, quadratic and linear ones. The current functions are linear in both cases.

For both cases, the main result is the outward displacement of the magnetic axis, corresponding to a degenerate flux surface of zero volume. However, the flux surfaces do not coincide with the constant pressure surfaces, and a centrifugally corrected pressure is a surface quantity in those cases. The axis displacement increases in a nonlinear fashion with the plasma angular velocity.

The vertical magnetic field has a reversal already in the static case, and the radius at which this reversal occurs is also outwardly displaced with increasing angular velocity. This effect is observed for both linear and quadratic profiles. Moreover, the vertical current density has essentially the same behavior.

A difference between linear and quadratic pressure profiles arises in the azimuthal component of the current density. In the quadratic case, we observe a reversal of it, but the reversal radius decreases with the angular velocity. By way of contrast, in the linear case the static configuration has no sign reversal, but the latter appears with increasing angular velocity. Moreover, while in the static case the current density vanishes at the boundary (a conducting drum), the rotation induces a nonzero value there. This feature is undesirable from the point of view of achieving a stable stationary equilibrium.

ACKNOWLEDGMENTS

The authors would like to express their thank for the useful discussions had with Professor Iberê Luiz Caldas (University of São Paulo). This work has been supported by grants from CNPq and CAPES (Brazilian Government Agencies).


