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Recurrence quantification analysis of chimera states

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ABSTRACT

Chimera states, characterised by coexistence of coherence and incoherence in coupled dynamical systems, have been found in various physical systems, such as mechanical oscillator networks and Josephson-junction arrays. We used recurrence plots to provide graphical representations of recurrent patterns and identify chimera states. Moreover, we show that recurrence plots can be used as a diagnostic of chimera states and also to identify the chimera collapse.

1. Introduction

Network dynamical systems have been studied as models of spatiotemporal complexity. Among the spatiotemporal features recognised in coupled systems we can find chaos synchronisation [1], suppression [2,3], pattern formation [4], and multistability [5].

Dynamical systems may be modelled by coupled ordinary differential equations (CODE). A network of coupled differential equations has a continuous state variable and time, while the space is discrete. CODE present various applications to spatially extended systems in nonlinear dynamical systems [6]. For instance, production and transfer of energy and information in conservative systems [7], creation of hyperchaotic attractors in a system of coupled Chua circuits [8], and phase synchronisation between collective rhythms of coupled oscillator groups [9]. Moreover, biophysical complex systems may be modelled by coupled differential equations, such as tumour growth [10,11], and synchronisation of bursting neurons [12,13].

Here we focus on dynamical features in CODE such as coherence and incoherence states. When these states coexist the phenomenon is called a chimera state [14]. The network contains a coherent and phase locked domain, and an incoherent domain. The coexistence of coherence and incoherence was first observed by Kuramoto and Battogtokh in a non-locally coupled phase oscillators [15]. The existence of chimera states has also been verified in networks with symmetrically coupled identical oscillators [16].

Recently, it has been shown that chimera states can be seen in experimental studies. Hagerstrom and coworkers showed that these states can be realised in experiments using a liquid-crystal spatial light modulator [17]. Tinsley and coworkers reported experimental studies in which they observed chimera states in coupled Belousov–Zhabotinsky oscillators [18]. In addition, an experimental work about chimera states can be found in Ref. [19], where it was shown that chimeras could emerge coupled mechanical oscillators. The experimental setup was realised with metronomes coupled by means of adjustable springs. Swing and metronome displacements were measured by digital tracking of UV fluorescent spots located on the pendula and swings. Through simple mechanical oscillators, known as Huygen clock, Kapitaniak and collaborators [20] verified the existence of imperfect chimera states in pendula coupled on the ring by means of springs and dampers.

Our main result is to show that recurrence quantification analysis can be used as a diagnostic of chimera states. Recurrence analysis is a graphical method designed to locate hidden recurring patterns, structural and non-stationarity changes [21,22]. Recurrence quantification can be applied to scientific data. Marwan and collaborators [23] applied recurrence analysis of time series to a marine palaeo-climate record. They identified the subtle changes to the climate regime. Recurrence quantification was also considered by Zbilut and collaborators [24] as a tool for nonlinear exploration of non-stationary cardiac signals. Ding analysed the combination of three recurrence quantification analysis variables [25]. Local complex recurrence plot structures were explored and the

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results demonstrated that the combination improved nonlinear dynamic discriminant analysis.

With regard to recurrence quantification, we have calculated recurrence rate, determinism, and laminarity when the system exhibits chimera states. In this work, we have verified that the recurrence quantification is a good diagnostic for the determination of chimera states, as well as for identification of the collapse of a chimera state.

This paper is organised as follows. Section 2 introduces the model equations. In Section 3, the plot of recurrence is proposed as a diagnostic for the identification of chimera states. In the last section, we draw the conclusions.

2. Chimera states

We consider a spatially extended system formed by coupled ordinary differential equations, in which the space is discrete, while the state variable and the time are continuous. The network to be treated in this work is a set of Kuramoto oscillators that can exhibit coherent and incoherent behaviours, and it is given by

$$\dot{\Psi}_k(t) = \omega_k - \frac{1}{2R}\sum_{j=k-R}^{k+R} \sin[\Psi_k(t) - \Psi_j(t) + \alpha], \tag{1}$$

where the system is composed of N oscillators, each oscillator k ($1 \leq k \leq N$) with phase Ψ_k has an intrinsic natural frequency ω_k, R is the coupling range, and α is Sakaguchi's phase lag parameter [26]. Several nontrivial synchronisations can be observed for certain phase lags, such as decreasing synchronisation with increasing coupling strength, coexistence of stable incoherence with a partially synchronised state, and coexistence of two stable partially synchronised state [27]. In our simulations we consider $r = R/N$, $\omega_k = 0$, and the initial conditions are distributed in the interval $[-\pi, \pi]$ aiming to obtain chimera states. For $\omega_k = \omega$, Abrams and Strogatz [28] had obtained chimera states for nonlocal coupled oscillators. Rosin and collaborators [29] studied a nonlocally network of coupled electronic oscillators that approximately follows a Kuramoto-like model. They assumed identical oscillators to observe chimera states, namely the same intrinsic natural frequency for all oscillators. Laing and collaborators showed that similar patterns occur with nearly identical oscillators [30]. In this article, we considered a finite range coupling (1) that can exhibit chimera states. Moreover, this coupling presents a local (next-neighbour) coupling when $R = 1$, and a global (all-to-all) coupling when $R = (N/2) - 1$ [31].

Fig. 1 displays the scenario of coherence and incoherence states. Space–time plots are showed in the left column, and snapshots in the right column for phase lag parameter equal to 1.57, 1.47, and 1.37. The dynamics is spatially incoherent in Fig. 1a and 1b for $\alpha = 1.57$. Decreasing the value of α for 1.47 we can observe chimera state (Fig. 1c), where the oscillators with indices from 5 to 30 are in an incoherent state, while the remaining oscillators are in a spatially coherent state (Fig. 1d). In Fig. 1e and 1f, for $\alpha = 1.37$, the dynamics is spatially coherent.

3. Recurrence quantification analysis

We have studied the recurrence plots as a diagnostic of chimera states. Recurrence plots was introduced by Eckmann and collaborators [32], and it is based on the visualisation of a square matrix. The matrix elements correspond to times at which a state recurs. In the case of time series, the recurrence plot shows when the time series visits the same region of the phase space. In our case, instead of time series we use the recurrence plot in spatial series, that is given by

$$\mathrm{RP}_{i,j} = \Theta(\varepsilon - \|\Psi_i - \Psi_j\|), \tag{2}$$

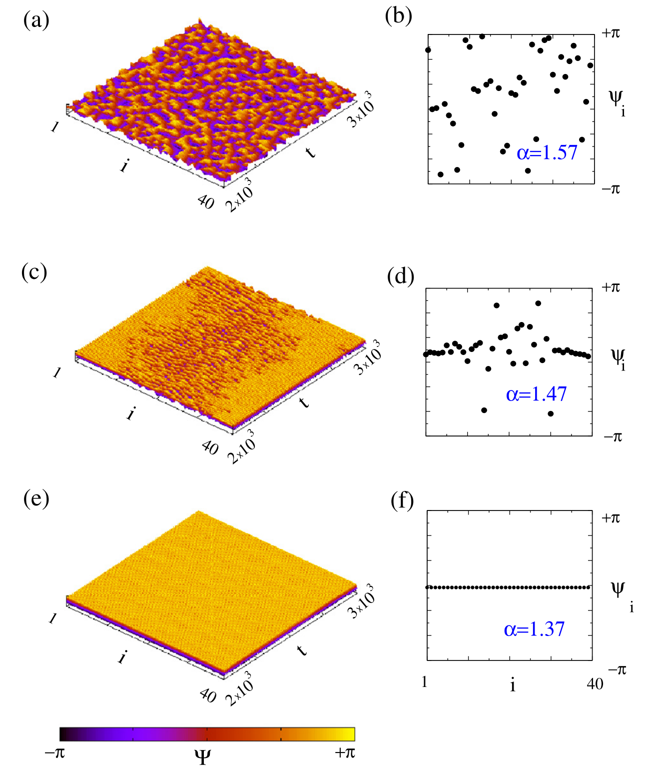

Fig. 1. (Colour online.) Space–time plots (left) and snapshots of the phases Ψ_i (right) for $r = 0.35$, $N = 40$, phase lag parameter equal to 1.57, 1.47, and 1.37. The chimera state in (c) and (d) results from a carefully chosen initial condition. The colour bar represents the values of Ψ_i.

where $\Psi_i \in \Re^m$ ($i, j = 1, \ldots, N$), N is the number of states Ψ_i, i and j in a m-dimensional space, ε is a threshold distance, $||.||$ stands for the Euclidean norm, and $\Theta(.)$ is the Heaviside function.

Fig. 2 shows recurrence plots for different values of the of the phase lag parameter and three different values of recurrence thresholds. In Figs. 2a, b, and c, we consider α equal to 1.57 for $\varepsilon = 0.01$, 0.1, and 0.3, respectively, the recurrence plot for the three cases shows one diagonal without large structures. When α is equal to 1.47 for a small ε value (Fig. 2d) there are few structures, and only some few sparse points. For an intermediate ε value the plot exhibits not only one diagonal line, but also large structures, as a result of coherent regions of a chimera state (Fig. 2e). The third case of $\alpha = 1.47$ (Fig. 2f) is for the biggest ε value. We observe a huge number of structures due the fact that an incoherent region is not anymore distinguished from coherent regions if we use an overestimated value of recurrence threshold. For α equal to 1.37 we can only see one grey region, that is independent of the ε value used (Figs. 2g, h, and i). The recurrence plot is completely grey due to regular spatial behaviour of the coupled oscillators. If we are interested in the quantification of the coherent regions observed in a chimera state, our results (Fig. 2) show that the intermediate value $\varepsilon = 0.1$ is optimal.

The recurrence quantification analysis can provide information about the system through the measures of complexity. A recurrence occurs whenever two states Ψ_i and Ψ_j visits roughly the same region in a m-dimensional space. For this reason, we have studied the chimera states that could be identified by means of the measures: recurrence rate (RR), determinism (DET), and laminarity (LAM) [33]. The recurrence rate (RR) is the density of recurrence point, given by

$$\mathrm{RR}(\varepsilon) = \frac{1}{N^2}\sum_{i,j=1}^{N} \mathrm{RP}_{i,j}(\varepsilon), \tag{3}$$

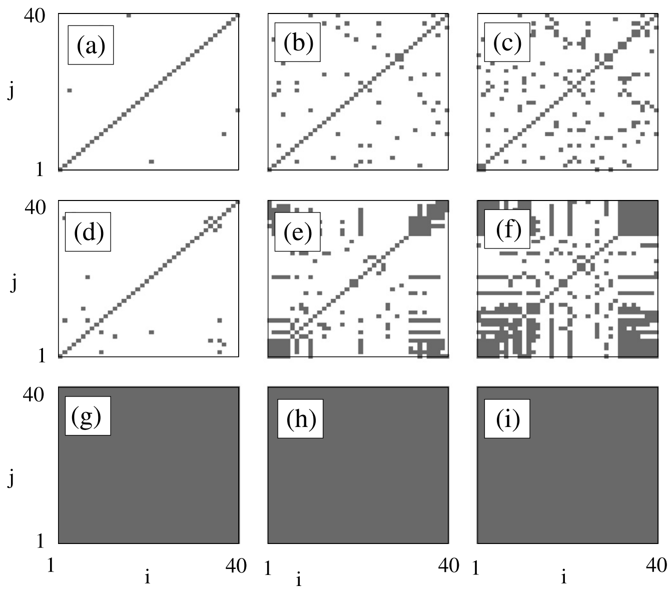

Fig. 2. Recurrence plots for $r = 0.35$, $N = 40$, $\alpha = 1.57$, and ε equal to (a) 0.01, (b) 0.1, and (c) 0.3, $\alpha = 1.47$, and ε equal to (d) 0.01, (e) 0.1, and (f) 0.3, $\alpha = 1.37$, and ε equal to (g) 0.01, (h) 0.1, and (i) 0.3.

which corresponds the rate between the grey recurrence points ($\mathrm{RP}_{i,j} = 1$) and the total number of pixels (grey and white) in a recurrence plot. The second measure is given by the number of diagonal lines in recurrence plot, which are structures parallels to the line of identity ($\mathrm{RP}_{i,i} = 1$, $i = 1, 2, \ldots, N$), and defined as

$$\mathrm{RP}_{i+k,j+k} = 1 \quad (k = 1, 2, \ldots, l), \tag{4}$$

$$\mathrm{RP}_{i,j} = \mathrm{RP}_{i+k+1,j+k+1} = 0, \tag{5}$$

where l is the length of a diagonal line. The determinism is the percentage of recurrence points that form diagonal lines,

$$\mathrm{DET} = \frac{\sum_{l=l_{\min}}^{N} l P(l)}{\sum_{l=1}^{N} l P(l)}, \tag{6}$$

where $P(l)$ is the frequency distribution of the lengths l of the diagonal lines, and $l_{\min}$ is a minimal length. The diagonal lines occurs when a segment in a spatial profile runs parallel to another segment. The third measure is related with the presence of vertical lines in the recurrence plot. The definition of a vertical line is

$$\mathrm{RP}_{i,j+k} = 1 \quad (k = 1, 2, \ldots, v), \tag{7}$$

$$\mathrm{RP}_{i,j} = \mathrm{RP}_{ij+v+1} = 0, \tag{8}$$

where v is the length of a vertical line. The laminarity is the percentage of points that form vertical lines,

$$\mathrm{LAM} = \frac{\sum_{v=v_{\min}}^{N} v P(v)}{\sum_{v=1}^{N} v P(v)}, \tag{9}$$

where $P(v)$ is the frequency distribution of the lengths v of the vertical lines, and $v_{\min}$ is a minimal vertical line. The vertical structures occurs in a recurrence plot when a spatial state remains equal or change very little. In particular, the laminarity will be useful in this work to distinguish the coherent and the incoherent in the chimera states.

We can see in Fig. 3 the recurrence quantification analysis for different values of the phase lag parameter. Fig. 3a, for $\alpha = 1.57$, shows that the laminarity and the determinism present irregular oscillations with values larger than the recurrence rate, that is the case for spatially incoherent state. We have plotted a histogram that shows the distribution of lengths of the vertical lines. For the spatially incoherent state the histogram presents one peak that is associated with a minimum value equal to one (Fig. 3b). When $\alpha = 1.47$ we have the chimera state, and we verify that the irregular oscillations of the laminarity and the determinism continue with values larger than the recurrence plot, as shown in Fig. 3c. However, in this case the values oscillate with amplitude larger than the spatially incoherent state. The histogram is skewed right, as showed in Fig. 3d. Decreasing the phase lag parameter for 1.37 the network exhibits spatially coherent state, as a result this behaviour we observe by means Fig. 3e which the recurrence rate, the laminarity, and the determinism have values equal to 1. Consequently, the histogram exhibits one peak at $N - 1$ (Fig. 3f). In addition, we calculate the global order parameter $Z(t) = \frac{1}{N}\left|\sum_{j=1}^{N} \exp(i\Psi_j(t))\right|$, where $Z = 1$ when the network presents completely synchronisation, and $Z \ll 1$ for uncorrelated behaviour. In Fig. 3, we can see that the global order parameter exhibits a small irregular oscillation for the spatially incoherent behaviour, irregular oscillation around 0.75 for chimera state, and value equal to 1 for the spatially coherent state. Therefore, in these situations the recurrence quantification analysis and the global order parameter present similar results.

In this work, we have verified that spatially incoherent state exhibits smaller values for the laminarity, determinism, and recurrence rate than coherent and chimera states. The coherent state presents approximately constant values for the laminarity, determinism, and recurrence rate. Therefore, the identification of chimera states is possible through recurrence quantification analysis, due to the fact that recurrence rate, laminarity, and determinism present not only larger values in the chimera states than in spatially incoherent state, but also the values are not constant as in spatially coherent state.

It is possible to observe the collapse of chimeras, namely the chimera states may disappear after a temporary stable existence. After the collapse the system modifies the behaviour from incoherent to a coherent. Wolfrum and Omel'chenko studied chimera states in identical non-locally coupled phase oscillators [34]. They verified that, for a small populations of oscillators, the chimera states sudden collapse after a certain time span. With this in mind, we consider a network with 40 and 100 oscillators to analyse the collapse through recurrence quantification. Fig. 4a and 4b exhibit the phase Ψ_i in colour scale according to Fig. 1. The time evolution shows a chimera state that disappears after a certain time. We can see the collapse of the chimera for a time approximately equal to 26 800 (Fig. 4a) and 2300 (Fig. 4b). In terms of recurrence plot this means a presence of some vertical structures before the collapse of the chimera. This can be observed in Fig. 4c and 4d with the recurrence plot using the probability distribution of the vertical lengths. Before the collapse, we have a positive non-zero value of the probability distribution of the vertical lengths, in other words, this means that we have the simultaneous coexistence of coherent and incoherent regions. After the collapse, as shown in Fig. 4e, the value of the distribution of the vertical length is the maximum value because the spatial synchronisation of the oscillators. As a result, the laminarity, the determinism, the recurrence rate, and global order parameter present constant values (equal to 1) after the collapse (Fig. 4e), due to the network arrives in a coherent state. Nevertheless, if the network arrives in a frequency synchronisation, the laminarity, the recurrence rate, and the global order parameter present small values, while only the determinism goes to a value equal to 1, as shown in Fig. 4f. Therefore, time series of the global order parameter series provide values less than one when the breakdown of the chimera is a state of a frequency synchronisation. Then, the recurrence quantification analysis is able to identify the collapse and distinguish when the

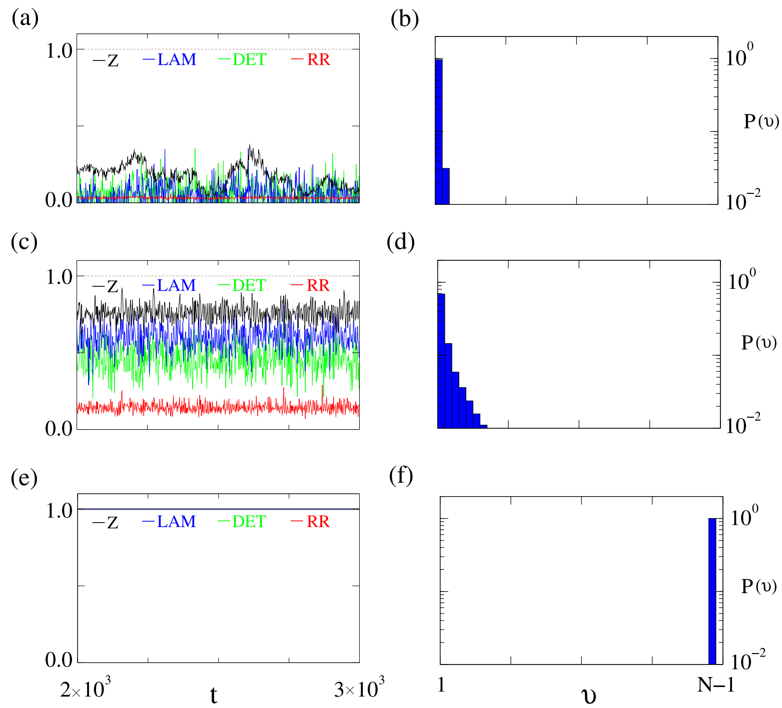

Fig. 3. (Colour online.) Recurrence rate (RR), laminarity (LAM), determinism (DET), and global order parameter (Z) for $r = 0.35$, $N = 40$, $\varepsilon = 0.1$, (a) $\alpha = 1.57$, (c) $\alpha = 1.47$, and (e) $\alpha = 1.37$. Histogram of lengths of the vertical lines for (b) $\alpha = 1.57$, (d) $\alpha = 1.47$, and (f) $\alpha = 1.37$.

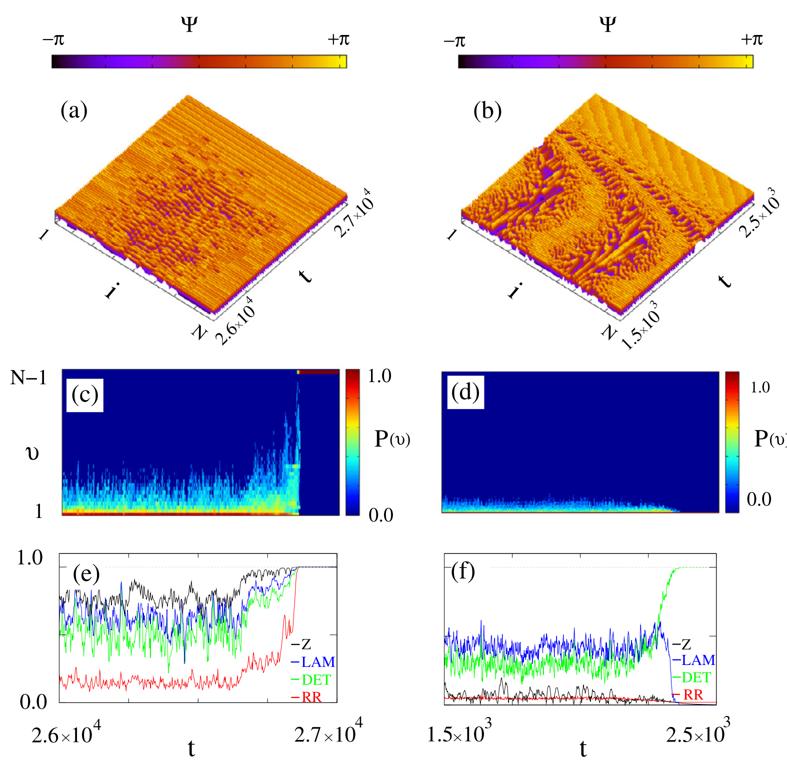

Fig. 4. (Colour online.) (a) and (b) $i \times t$ where the phase Ψ_i are coloured according to the colour bar of Fig. 1, respectively, for $r = 0.35$, $\alpha = 1.47$ and $N = 40$ (left column), $r = 0.25$, $\alpha = 1.47$ and $N = 100$ (right column). (c) and (d) temporal evolution of the probability distribution of vertical lengths. (e) and (f) recurrence rate (RR), determinism (DET), laminarity (LAM), and global order parameter (Z) for $\varepsilon = 0.1$.

final state is the spatial synchronisation, or the frequency synchronisation.

We compute the average lifetime T of the chimera states varying the network size N. Wolfrum and collaborators observed an exponential growth $T \sim \exp(kN)$, where k is the exponential rate, detecting the collapse by means of the global mean field [34]. We obtain the average lifetime through recurrence analysis. According to Fig. 4c, the average lifetime can be computed by means of the time that orbits, starting from a set of initial conditions, spend before the values of recurrence rate, determinism, and laminarity are equal to one reach. We fix the value of N, and compute the mean of the average lifetime for a set of 2000 random initial conditions

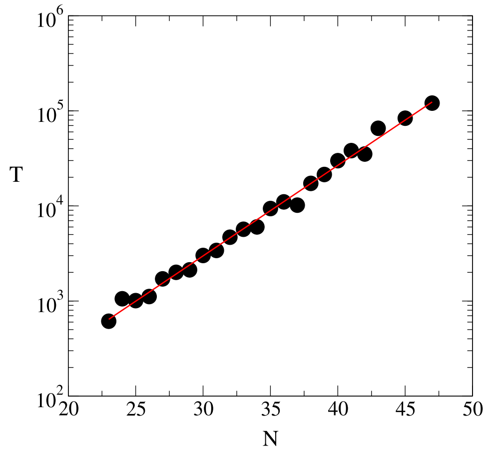

Fig. 5. (Colour online.) Average collapse time of chimera states as a function of the network size for $r = 0.35$, $\alpha = 1.47$ and from 2000 trajectories, where the black circles are obtained from simulation and the red line is the fitted exponential.

slightly disturbed from the trajectory, as shown in Fig. 4a. Fig. 5 shows the average lifetime of chimera states as a function of the network size, where the black circles are obtained from simulation and the red line is the fitted exponential growth given by $T = 4.09\exp(0.22N)$. As expected the lifetime increases exponentially with the network size [34], with a characteristic exponential rate $k = 0.22$.

4. Conclusions

Since the discovery of chimera states, numerous studies have been realised about the evolution of the chimera state using the global and local order parameter as diagnostic tool. The order parameter presents value close to unity for the coherent state and decreases in spatial incoherence domains. Before the chimera collapse the incoherent regions are detected using local order parameter, and after the collapse the network presents only coherent regions identified when the global order parameter value is equal one. In this work, we show that recurrence quantification analysis is also an useful tool as diagnostic not only to identify chimera states, but also to determine the chimera collapse. The quantification of recurrence plots is a powerful tool that can also provide the degree of determinism, state changes, and degrees of complexity of systems.

In conclusion, we have shown that recurrence quantification analysis can be used as a diagnostic of chimera states. The difference between chimera states and spatially incoherent states is showed by laminarity, determinism, and recurrence rate. The chimera states present values larger than spatially incoherent states. However, the difference between chimera states and coherent states are observed through oscillations in the values computed for the chimera states. The histogram of lengths of the vertical lines obtained from recurrence plots is skewed right for the chimera states. Moreover, the recurrence quantification analysis can be used to identify the collapse of chimera states. After the collapse of the chimera we can see that the laminarity, the determinism, and the recurrence rate present constant values. The recurrence quantification is a good diagnostic tool to identify the chimera collapse when the final state is not only the spatial synchronisation, but also the frequency synchronisation. In the frequency synchronisation, the network presents a high determinism while the laminarity and recurrence rate are low. We also verified that the average life time of chimera states increases exponentially as a function of the size network.

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References

[1] S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, Phys. Rep. 366 (2002) 1.
[2] L. Molgedey, J. Schuchhardt, H.G. Schuster, Phys. Rev. Lett. 69 (1992) 3717.
[3] M.A. Matías, J. Güémez, Phys. Rev. E 54 (1996) 198.
[4] P.G. Lind, J. Corte-Real, J.A.C. Gallas, Phys. Rev. E 69 (2004) 066206.
[5] M.S.T. Freitas, R.L. Viana, C. Grebogi, Int. J. Bifurc. Chaos Appl. Sci. Eng. 14 (2004) 927.
[6] C. Chicone, Ordinary Differential Equations with Applications, second ed., Springer Science and Business Media, 2006.
[7] C.G. Antonopoulos, E. Bianco-Martinez, M.S. Baptista, PLoS ONE 9 (2014) e89585.
[8] V.S. Anishchenko, T. Kapitaniak, M.A. Safonova, O.V. Sosnovzeva, Phys. Rev. Lett. 192 (1994) 207.
[9] Y. Kawamura, H. Nakao, K. Arai, H. Kori, Y. Kuramoto, Chaos (2010) 043110.
[10] F.S. Borges, K.C. Iarosz, H.P. Ren, A.M. Batista, M.S. Baptista, R.L. Viana, S.R. Lopes, C. Grebogi, Biosystems 116 (2014) 43.
[11] K.C. Iarosz, F.S. Borges, A.M. Batista, R.A.N. Siqueira, M.S. Baptista, R.L. Viana, S.R. Lopes, J. Theor. Biol. 368 (2015) 113.
[12] C.A.S. Batista, R.L. Viana, F.A.S. Ferrari, S.R. Lopes, A.M. Batista, J.C.P. Coninck, Phys. Rev. E 87 (2013) 042713.
[13] C.A.S. Batista, R.L. Viana, S.R. Lopes, A.M. Batista, Physica A 410 (2014) 628.
[14] Y. Zhu, Z. Zheng, J. Yang, Phys. Rev. E 89 (2014) 022914.
[15] Y. Kuramoto, D. Battogtokh, Nonlinear Phenom. Complex Syst. 5 (2002) 380.
[16] D.M. Abrams, R. Mirollo, S.H. Strogatz, D.A. Wiley, Phys. Rev. Lett. 101 (2008) 084103.
[17] A.M. Hagerstrom, T.E. Murphy, R. Roy, P. Hövel, I. Omelchenko, E. Schöll, Nat. Phys. 8 (2012) 658.
[18] M.R. Tinsley, S. Nkomo, K. Showalter, Nat. Phys. 8 (2012) 662.
[19] E.A. Martens, S. Thutupalli, A. Fourrière, O. Hallatschek, Chimera states in mechanical oscillator networks, Proc. Natl. Acad. Sci. USA 110 (26) (2013) 10563–10567.
[20] T. Kapitaniak, P. Kuzma, J. Wojewoda, K. Czolczynski, Y. Maistrenko, Imperfect chimera states for coupled pendula, Sci. Rep. 4 (2014) 6379.
[21] D.B. Vasconcelos, R.L. Viana, S.R. Lopes, J. Kurths, Phys. Rev. E 73 (2006) 0562076.
[22] T.L. Prado, P.P. Galuzio, S.R. Lopes, R.L. Viana, Chaos 24 (2014) 013106.
[23] N. Marwan, J.F. Donges, Y. Zou, R.V. Donner, J. Kurths, Phys. Lett. A 373 (2009) 4246.
[24] J.P. Zbilut, N. Thomasson, C.L. Webber, Med. Eng. Phys. 24 (2002) 53.
[25] H. Ding, Phys. Rev. E 89 (2014) 013313.
[26] H. Sakaguchi, Y. Kuramoto, Prog. Theor. Phys. 76 (1986) 576.
[27] O.E. Omel'chenko, M. Wolfrum, Phys. Rev. Lett. 109 (2012) 164101.
[28] D.M. Abrams, S.H. Strogatz, Phys. Rev. Lett. 93 (2004) 174102.
[29] D.P. Rosin, D. Rontani, N.D. Haynes, E. Schöll, D.J. Gauthier, Phys. Rev. E 90 (2014) 030902.
[30] C.R. Laing, K. Rajendran, I.G. Kevrekidis, Chimeras in random non-complete networks of phase oscillators, Chaos 22 (1) (2012) 013132.
[31] R. Gopal, V.K. Chandrasekar, A. Venkatesan, M. Lakshmanan, Phys. Rev. E 89 (2014) 052914.
[32] J.P. Eckmann, S.O. Kamphorst, D. Ruelle, Europhys. Lett. 5 (1987) 973.
[33] N. Marwan, M.C. Romano, M. Thiel, J. Kurths, Phys. Rep. 438 (2007) 237.
[34] M. Wolfrum, O.E. Omel'chenko, Phys. Rev. E 84 (2011) 015201.