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ORIGINAL PAPER

Dynamical changes from harmonic vibrations of a limited power supply driving a Duffing oscillator

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Abstract The harmonic oscillations of a Duffing oscillator driven by a limited power supply are investigated as a function of the alternative strength of the rotor. The semi-trivial and non-trivial solutions are derived. We examine the stability of these solutions and then explore the complex behaviors associated with the bifurcations sequences. Interestingly, a 3D diagram provides a global view of the effects of alternate strength on the appearance of chaos and hyperchaos on the system.

Keywords Non-ideal system · Averaging method · Lyapunov spectrum · Bifurcation diagram

1 Introduction

Much of the interest in non-linear dynamical systems has focused on the existence of periodic, quasiperiodic and chaotic attractors, and the investigation of how these arise [1–6]. Several recent studies have addressed the behaviors of oscillator driven by a limited power supply, such that the source of forcing is considered to be another oscillator, coupled to the first one [7–11]. As a particularity of these systems, called non-ideal oscillators, the forcing source has a limited available energy supply and the excitation is influenced by the system's response. Various dynamical phenomena, like periodic and chaotic motions, bifurcations, sudden changes in attractors, intermittency and their basins of attraction have been investigated in these systems (see Refs. [1–11]). Generally, these papers always consider a continuous voltage source strength. In this paper, we aim to analyze some dynamical features of the non-ideal model in a general context of non-linear dynamics, with special emphasis on the Duffing oscillator driven by a limited power supply [7, 8] while assuming an alternative source voltage. Taken into account the periodicity of the voltage source, we focus on the existence of trivial and non-trivial solutions along with the quenching phenomenon. The effects of alternate strength on the stability of oscillation are also analyzed. The paper is organized as follows: Section 2 introduces the combined system of ordinary and algebraic equations modeling the oscillator and its related dynamical behavior using averaging methods similar to the one presented in Warminsky [12]. In Sect. 3, the derivation of semi-trivial solution associated with the quenching of amplitude of vibration is given along with the stability conditions. Section 4 fo-

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cuses on the non-trivial solution. In this case, some bifurcation structures together with Lyapunov spectrum are derived and the effects of alternate strength of voltage source are pointed out. Our conclusion is left to the final section.

2 Model and harmonic oscillatory states

The model (see Fig. 1) represents the one-dimensional motion of a cart connected to a fixed frame by a non-linear spring and a dashpot [7]. The motion of the cart is due to an in-board non-ideal motor with driving and unbalanced rotor. We denote by φ the angular displacement of the rotor modeled as a particle of mass attached to a massless rod with respect to the rotation axis (see Ref. [11]). x denotes the cart displacement with respect to some equilibrium position in the absolute reference frame. The following equations give the motion of the system [7, 11]:

$$\begin{aligned} &\ddot{x} + \beta\dot{x} - x + \delta x^3 = \epsilon_1\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right), \\ &\ddot{\varphi} = \epsilon_2\ddot{x}\sin\varphi + E_1 - E_2\dot{\varphi}. \end{aligned} \tag{1}$$

β, δ are, respectively, the damping coefficient and the non-linear parameter of the potential. ϵ_1 and ϵ_2 are the physical characteristics of the system [8]. E_1 and E_2 are, respectively, the voltage or the strength of the motor and the characteristic parameter of the considered motor. The dots stand for differentiation with respect to scaled time t. It has been shown that this type of system vibrates under the principal parametric resonance condition 2 : 1 [12, 13]. Taking into account this assertion we assume that the oscillator vibrates with frequency ω, but the frequency of the non-ideal force is equal to $\frac{\omega}{2}$. Thus, considering that the voltage source is alternated means that E_1 is periodic and can be written in this form:

$$E_1 = u_0\cos\frac{\omega}{2}t, \tag{2}$$

where u_0 is the amplitude of the voltage source. Equations (1) become

$$\begin{aligned} &\ddot{x} + \beta\dot{x} - x + \delta x^3 = \epsilon_1\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right), \\ &\ddot{\varphi} = \epsilon_2\ddot{x}\sin\varphi + u_0\cos\frac{\omega}{2}t - E_2\dot{\varphi}. \end{aligned} \tag{3}$$

Due to the strong non-linearity of this system, there are no exact analytical solutions for Eqs. (3). However, to obtain an analytical approach of these solutions, we use an averaging method, similar to the one presented by Warminsky [12], and taking into account the internal resonance 2 : 1, we set

$$\begin{aligned} &x(t) = A_0 + A_1(t)\cos\omega t + A_2(t)\sin\omega t, \\ &\varphi(t) = B_0 + B_1(t)\cos\frac{\omega}{2}t + B_2(t)\sin\frac{\omega}{2}t, \end{aligned} \tag{4}$$

where A_0 and B_0 are, respectively, the amplitude of the structure and the rotor at rest. $A_1(t)$, $A_2(t)$, $B_1(t)$, and $B_2(t)$ are functions slowly changing in time that represent the oscillator's and rotor amplitudes which are defined as $A = \sqrt{A_1^2 + A_2^2}$ and $B = \sqrt{B_1^2 + B_2^2}$, respectively. To obtain approximated solutions, we expand the non-linear function $\sin\varphi$ and $\cos\varphi$ in the Taylor series until third order around the lower steady state for $\varphi = 0$. Thus, substituting (4) into (3) and balancing the harmonics, yield these set of first order approximate differential equations:

$$\begin{aligned} &\beta\dot{A}_1 + 2\omega\dot{A}_2 - \omega\epsilon_1 B_2\dot{B}_1\left(1 - \frac{B_1^2}{4} - \frac{B_2^2}{12} + \frac{5}{4}B_0^2\right) - \omega\epsilon_1 B_1\dot{B}_2\left(1 - \frac{B_1^2}{4} - \frac{B_2^2}{12} + \frac{5}{4}B_0^2\right) \\ &\quad + \left(-1 - \omega^2 + \frac{3}{4}\delta\left(A_1^2 + A_2^2\right) + 3\delta A_0^2\right)A_1 + \beta\omega A_2 + \frac{\omega^2}{4}\epsilon_1\left(B_1^2 - B_2^2\right) - \frac{\omega^2}{8}\epsilon_1 B_0^2\left(B_1^2 - B_2^2\right) = 0, \\ &\beta\dot{A}_2 - 2\omega A_1 + \omega\epsilon_1 B_1\dot{B}_1\left(1 - \frac{B_1^2}{6} - \frac{1}{2}B_0^2\right) - \omega\epsilon_1 B_2\dot{B}_2\left(1 - \frac{B_2^2}{6} - \frac{1}{2}B_0^2\right) \\ &\quad + \left(-1 - \omega^2 + \frac{3}{4}\delta\left(A_1^2 + A_2^2\right) + 3\delta A_0^2\right)A_2 - \beta\omega A_1 + \frac{\omega^2}{2}\epsilon_1 B_1 B_2 - \frac{\omega^2}{4}\epsilon_1 B_1 B_2 B_0^2 = 0, \end{aligned}$$

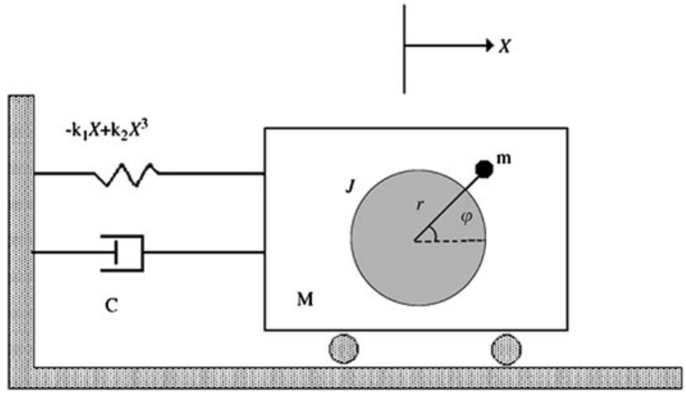

Fig. 1 Schematic description of a non-ideal model

$$
\begin{aligned}
&\omega\epsilon_2 B_2\dot{A}_1\left(1-\frac{B_1^2}{4}-\frac{B_2^2}{12}-\frac{B_0^2}{2}\right)+\omega\epsilon_2 B_1\dot{A}_2\left(1-\frac{B_0^2}{2}-\frac{B_1^2}{6}\right)+E_2\dot{B}_1+\omega\dot{B}_2+\frac{\omega}{2}E_2B_1\\
&\quad-\frac{\omega^2}{4}B_1-u_0+\frac{\omega^2}{2}\epsilon_2A_1B_1\left(1-\frac{B_0^2}{2}-\frac{B_1^2}{6}\right)+\frac{\omega^2}{2}\epsilon_2B_2\dot{A}_2\left(1-\frac{B_0^2}{2}-\frac{B_1^2}{4}-\frac{B_2^2}{12}\right)=0,\\
&\omega\epsilon_2 B_1\dot{A}_1\left(1-\frac{B_0^2}{2}-\frac{B_2^2}{4}-\frac{B_1^2}{12}\right)-\omega\epsilon_2 B_2\dot{A}_2\left(1+\frac{B_0^2}{2}+\frac{B_1^2}{6}\right)+E_2\dot{B}_2-\omega\dot{B}_1-\frac{\omega}{2}E_2B_1\\
&\quad-\frac{\omega^2}{4}B_2+\frac{\omega^2}{2}\epsilon_2A_1B_2\left(-1+\frac{B_0^2}{2}-\frac{B_2^2}{6}\right)+\frac{\omega^2}{2}\epsilon_2B_1A_2\left(1-\frac{B_0^2}{2}-\frac{B_1^2}{4}-\frac{B_2^1}{12}\right)=0,\\
&A_0\left(\frac{3}{2}\delta\left(A_1^2+A_2^2\right)+\delta A_0^2-1\right)=0,\\
&\frac{\omega^2}{8}\epsilon_2B_0\left(A_1\left(B_2^2-B_1^2\right)-2A_2B_1B_2\right)=0.
\end{aligned}
\tag{5}
$$

We have neglected the derivatives of the second order and terms having derivatives in a power higher than one. For steady states we have

$$\dot{A}_1=0,\quad \dot{A}_2=0,\quad \dot{B}_1=0\quad\text{and}\quad \dot{B}_2=0.$$

Moreover, for small oscillation we assume $B^2\approx 0$ and after some algebra, the following assertion is obtained:

$$
\begin{aligned}
&\delta A_0^2=1-\frac{3}{2}\delta A,\\
&B_0=0.
\end{aligned}
\tag{6}
$$

Equations (5) are thus reduced into a set of fourth nonlinear equations given by

$$
\begin{aligned}
&\left(-1-\omega^2+\frac{3}{4}\delta\left(A_1^2+A_2^2\right)+3\delta A_0^2\right)A_1\\
&\quad+\beta\omega A_2+\frac{\omega^2}{4}\epsilon_1\left(B_1^2-B_2^2\right)=0,\\
&\left(-1-\omega^2+\frac{3}{4}\delta\left(A_1^2+A_2^2\right)+3\delta A_0^2\right)A_2\\
&\quad-\beta\omega A_1+\frac{\omega^2}{2}\epsilon_1B_1B_2=0,\\
&\frac{\omega}{2}E_2B_2-\frac{\omega^2}{4}B_1+\frac{\omega^2}{2}\epsilon_2(A_1B_1+A_2B_2)\\
&\quad-\frac{\omega^2}{8}\epsilon_2A_2B_2B_1^2=u_0,\\
&-\frac{\omega}{2}E_2B_1-\frac{\omega^2}{4}B_2+\frac{\omega^2}{2}\epsilon_2(A_2B_1-A_1B_2)\\
&\quad-\frac{\omega^2}{8}\epsilon_2A_2B_2^2B_1=0.
\end{aligned}
\tag{7}
$$

The above algebraic equations may have semi-trivial and non-trivial solutions according to some specific considerations.

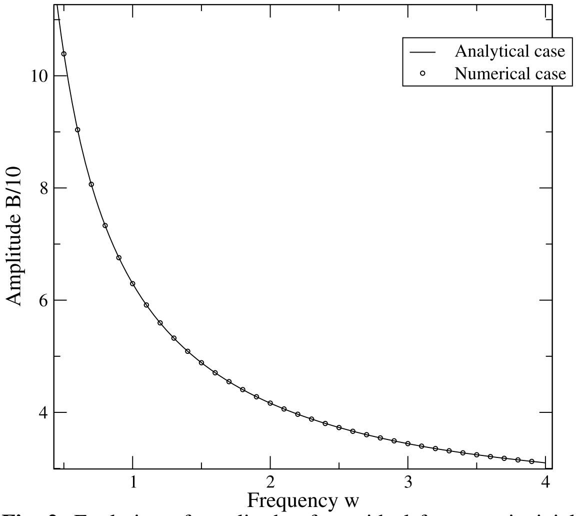

Fig. 2 Evolution of amplitude of non-ideal for a semi-trivial case

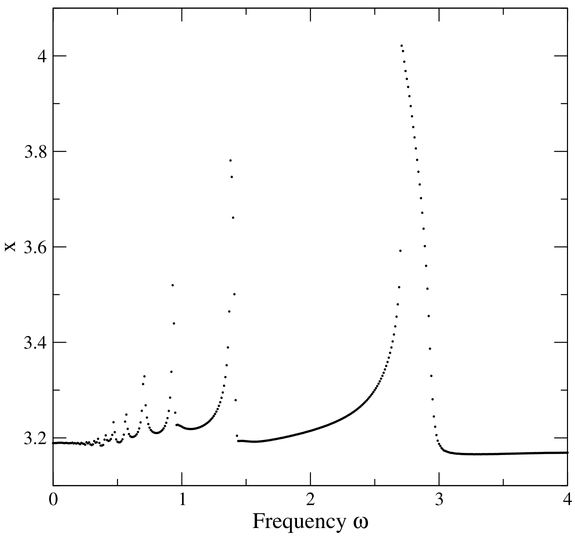

Fig. 3 Evolution of the amplitude of the mechanical part as function of frequency ω

3 Semi-trivial solutions and their stability

The semi-trivial solutions mean physically that one part of the system is oscillating while the other one is at rest. This gives the condition for quenching of amplitude of oscillation of one part of the system. We remind the reader that we use the following set of parameters for numerical purposes: $\beta = 0.02$, $\delta = 0.1$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.25$ and $E_2 = 1.5$ (see Ref. [9] for the choice of parameters).

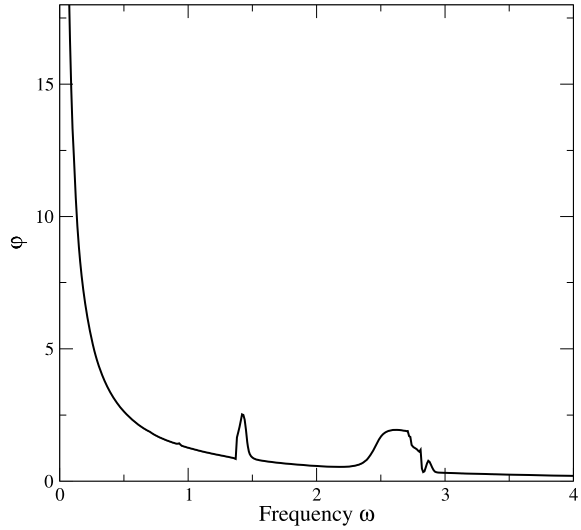

Fig. 4 Evolution of the amplitude of the non-ideal force as function of frequency

3.1 Quenching of amplitude in the mechanical part: $A_1 = A_2 = 0$ and $B_1 \neq 0$ and $B_2 \neq 0$

This situation represents the case where the mechanical part of the system do not vibrate. This is interesting since it can be viewed as a technique of control whose objective is to cancel the vibration of the mechanical system. Indeed, focusing on (5), the condition for quenching phenomenon in the space of parameters of the system is derived and given by

$$B = \frac{2u_0}{\omega\sqrt{E_2^2 + (\frac{\omega}{2})^2}}. \tag{8}$$

Figure 2 displays the evolution of the amplitude of non-ideal force B as function of the frequency ω given by (6) along with the one obtained by numerical simulations of the base equations (Eqs. (3)) using the fourth-order Runge–Kutta algorithm. It appears that the amplitude of the non-ideal force decreases as the frequency increases. This is confirmed by the results of numerical simulation (see dotted line). To analyze the stability of this semi-trivial solution we use the amplitude modulation equation given by

$$\dot{A}_1 = f_1(A_1, A_2, B_1, B_2),$$
$$\dot{A}_2 = f_2(A_1, A_2, B_1, B_2),$$
$$\dot{B}_1 = f_3(A_1, A_2, B_1, B_2),$$
$$\dot{B}_2 = f_4(A_1, A_2, B_1, B_2).$$

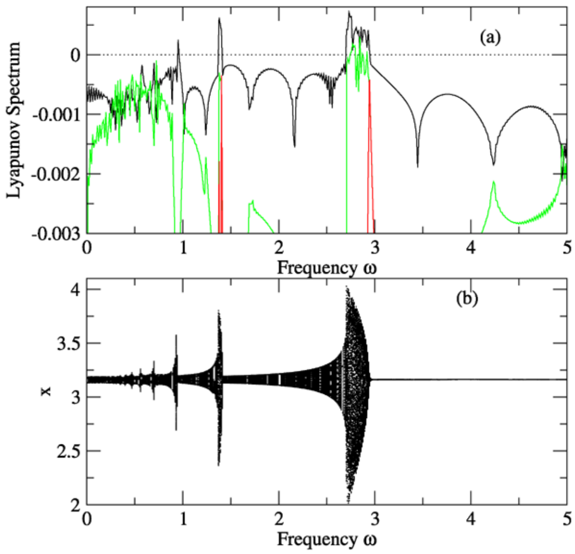

Fig. 5 Lyapunov spectrum and corresponding bifurcation diagram as function of ω for $u_0 = 1$

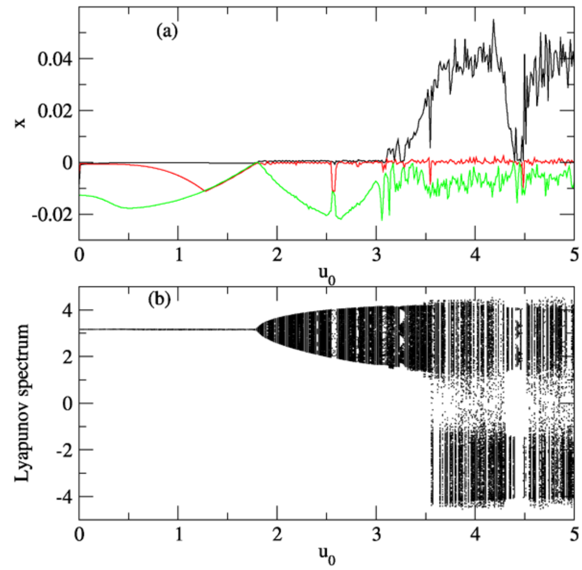

Fig. 6 Lyapunov spectrum and corresponding bifurcation diagram as function of u_0 for $\omega = 1.5$

Perturbing the analyzed solution by $A_1 + \delta A_1$, $A_2 + \delta A_2$, $B_1 + \delta B_1$, $B_2 + \delta B_2$, and getting a set of linear differential equations, we derive the stability conditions by obtaining the eigenvalues of the Jacobian [12]. For the above, which possesses a complex solution, the real part of the eigenvalue is always negative, stipulating that the system is always stable.

3.2 Quenching of amplitude of the non-ideal force: $B_1 = B_2 = 0$ and $A_1 \neq 0$ and $A_2 \neq 0$

This situation represents the case where the non-ideal forces does not swing and the structure vibrate. Using the set of differential equations (7) and assuming the stationarity of solutions leads after some algebraic to the following non-linear equation of the amplitude:

$$A = 0,$$
$$A^2 = \frac{(\frac{15}{4}\delta)^2 + \beta^2\omega^2 + (2-\omega^2)^2}{\frac{15}{2}\delta(2-\omega^2)} \tag{9}$$

where $A^2 = A_1^2 + A_2^2$. Analysis of this equation shows evidence that the mechanical part will stay at equilibrium and thus the system will remain stable.

4 Non-trivial solutions and their stability

The non-trivial solutions represent the case where both the systems vibrate. This means that $B_1 \neq 0$, $B_2 \neq 0$, $A_1 \neq 0$ and $A_2 \neq 0$. Analytically, we were supposed to use the set of equations (5) to derive the amplitude of both systems. Unfortunately, the system is strongly non-linear and it is quite impossible to obtain an analytical expression of amplitude. Moreover, to deal with such a question, we solved directly the base equations (Eqs. (3)) numerically using the fourth-order Runge–Kutta algorithm and discuss the amplitude response curves. Afterwards, we explore the stability of the system using Lyapunov spectrum and bifurcation sequences. The frequency response curve is thus obtained from (3) and presented in Fig. 3 for the evolution of x and in Fig. 4 for the evolution φ for $u_0 = 1$. Figure 3 reveals a set of subharmonic resonances instead of the internal resonance 2 : 1 as expected. Concerning the amplitude of non-ideal forces, it decreases as frequency increases and the effect of internal resonance if visible by their appearance.

Focusing on the stability of the system allows one to display the Lyapunov spectrum as function of frequency for the specify value of $u_0 = 1$. It appears for example in Fig. 5(a) that the Lyapunov spectrum is positive for certain values of frequency characterizing the presence of chaos in the system. On the other

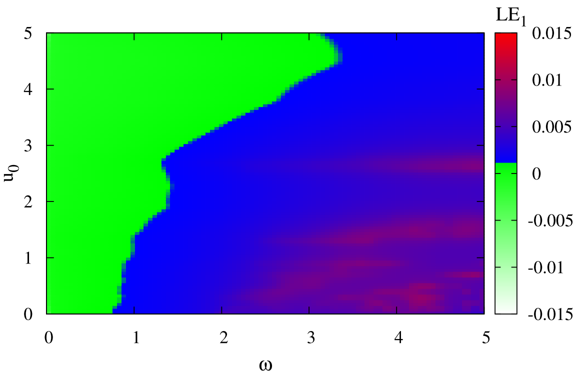

Fig. 7 Maximum Lyapunov exponent (LE_1) as function of u_0 and ω (Color figure online)

hand, Fig. 5(a) shows for another set of frequency more than one positive Lyapunov exponent, indicating hyperchaos on the system. These effects are confirmed via the corresponding bifurcation diagram (see Fig. 5(b)). Now setting $\omega = 1.5$ allows one to observe the stability of the system as u_0 increases. It appears in Fig. 6(a) that the system shows a transition from periodicity to quasiperiodicity and later on to chaos. To view how these transitions arise, we present the corresponding bifurcation diagram (see Fig. 6(b)), which confirm the results obtained from the Lyapunov spectrum and shows that for $u_0 = 0.36$, there is a crisis [14] in the system characterizing sudden changes in chaotic attractors. All this rich dynamics is summarized in Fig. 7 where a global view of the characteristics of alternate strength on Lyapunov spectrum in the form of 3D plot is displayed. It should be noted that this figure presents the maximum Lyapunov exponent with three associated colorbars showing different regions, indicating that Lyapunov values on increasing from minimum (green) to maximum (red) are clearly identified. The green colors denote regions in which the Lyapunov exponent is less or equal to zero, for periodic and quasiperiodic orbits. Increasing from blue to red, we have chaotic orbits. Explorations of the Lyapunov spectrum also show that there are some regions in which there is more than one positive Lyapunov spectrum indicating the presence of hyperchaos for a specific set of parameters. The regions marked with stars in the space parameter (ω, u_o) (see Fig. 8) represent the area of hyperchaos. This mean that when the voltage is taken to be periodic one should know that the characteristics of this source have a significant effects on the stability of the system.

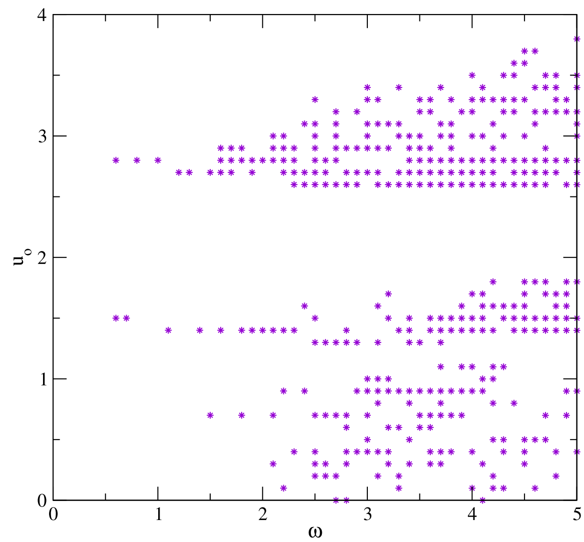

Fig. 8 Region in parameter space of (u_0, ω) where hyperchaos is detected

5 Conclusion

In this paper, we have investigated the dynamics of oscillator with limited power supply subjected to alternating strength of the voltage source. Considering the principal parametric resonance condition we derived in parameter space of the system the condition for the appearance of quenching phenomenons along with their stability. The non-trivial solutions have revealed an explosion of resonances 2 : 1 to a set of subharmonic resonance. The explorations of the stability of non-trivial solutions have led us to conclude that the periodicity of the voltage source perturbed the limited power supply and thus increase the possibility of the appearance of chaos and hyperchaos on the system. Consequently, when the voltage source is alternated, one should take into account the fact that for a bad choice of their characteristics the system can become unstable.

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