Parametric evolution of unstable dimension variability in coupled piecewise-linear chaotic maps

R. F. Pereira,[1] R. L. Viana,[2] S. R. Lopes,[2] M. C. Vergès,[3] and S. E. de S. Pinto[1,*]
[1]*Departamento de Física, Universidade Estadual de Ponta Grossa, 84030-900, Ponta Grossa, Paraná, Brazil*
[2]*Departamento de Física, Universidade Federal do Paraná, 81531-990, Curitiba, Paraná, Brazil*
[3]*Universidade Tecnológica Federal do Paraná, 84016-210, Ponta Grossa, Paraná, Brasil*
(Received 29 November 2010; published 24 March 2011)

In the presence of unstable dimension variability numerical solutions of chaotic systems are valid only for short periods of observation. For this reason, analytical results for systems that exhibit this phenomenon are needed. Aiming to go one step further in obtaining such results, we study the parametric evolution of unstable dimension variability in two coupled bungalow maps. Each of these maps presents intervals of linearity that define Markov partitions, which are recovered for the coupled system in the case of synchronization. Using such partitions we find exact results for the onset of unstable dimension variability and for contrast measure, which quantifies the intensity of the phenomenon in terms of the stability of the periodic orbits embedded in the synchronization subspace.

DOI: 10.1103/PhysRevE.83.037201 PACS number(s): 05.45.Xt, 05.45.Ra

Unstable dimension variability (UDV) is a form of nonhyperbolicity in which there is no continuous splitting between stable and unstable subspaces along the chaotic invariant set [1]. The variability takes place when the periodic orbits, embedded in the chaotic set, have a different number of unstable directions. This is a local phenomenon that can influence the entire phase space, and create complexity in the system [2–4]. The validity of trajectories generated by chaotic systems exhibiting UDV is guaranteed for short periods [5], which decreases as the intensity of the UDV increases [6,7].

The intensity of the UDV can be quantified by the embedded unstable periodic orbits (UPOs) in a nonhyperbolic attractor [3]. There are efficient computational methods for the analysis of these orbits [8,9]. However, it is a time-consuming task because the number of orbits increases with their period, and in many problems it is necessary to consider very high periods [10–12]. To avoid this problem, one constructs a model so that the UDV occurs in a transversal direction to a hyperbolic attractor. The dynamics in this attractor is well known, and therefore, some analytical results can be obtained. This type of construction allows us developing tools to shed light on the UDV [13,14]. Examples of physical problems that can be handled by these tools are the effect of shadowing in the kicked double rotor [15,16], the beginning of the spatial activity in the three-waves model [17,18], transport properties of passive inertial particles incompressible flows [19], and the chaos synchronization in coupled map lattices [20–22]. In some cases, the study of periodic orbits embedded in the synchronization subspace allows the global behavior determination of coupled chaotic maps [23].

The lack of accurate results hinders the understanding of the UDV. Thus, the key question that this article will address is the analytical calculation for systems that present such phenomenon. In the following pages, we shall consider a simple spatially extended system composed by two identical bungalow maps [24], which are piecewise linear and interact by a diffusive coupling. Such a system exhibits chaos synchronization and UDV in the transversal direction to the synchronization subspace, for certain parameters intervals [25]. Besides, this map presents strong chaos for the entire parameter control interval [26]. These features allows us to study the parametric evolution of the UDV for arbitrary periods.

Now, we shall consider the above-mentioned map $x \mapsto f_a(x)$, given by

$$f_a(x) = \begin{cases} \frac{1-a}{a}x & \text{if} \quad x \in I_1 \equiv [0,a), \\ \frac{2a}{1-2a}x + \frac{1-3a}{1-2a} & \text{if} \quad x \in I_2 \equiv \left[a,\frac{1}{2}\right), \\ \frac{2a}{1-2a}(1-x) + \frac{1-3a}{1-2a} & \text{if} \quad x \in I_3 \equiv \left[\frac{1}{2},1-a\right), \\ \frac{1-a}{a}(1-x) & \text{if} \quad x \in I_4 \equiv [1-a,1], \end{cases} \tag{1}$$

in which $a \in (0,1/2)$ is a parameter [see Fig. 1(a)]. This map has the following property [24]: $\forall a$, the four intervals of linearity of the map define four Markov partitions[1] of phase space $\omega = \bigcup I_\alpha = [0,1]$ (going forward, greek indexes range from 1 to 4). This property, consequently, allows us to study the symbolic dynamics and the interval dynamics of the system. Therefore, we hypothesize that the Markov partitions allow an exact result for the onset of the UDV in the synchronization subspace of coupled bungalow maps.

To do this study, we must determine all possible itineraries. Considering the linearity of the map in each interval I_α and the images of its ends

$$\left.\begin{aligned} f_a(0) &= 0 \\ f_a(a) &= 1-a \\ f_a(1/2) &= 1 \\ f_a(1-a) &= 1-a \\ f_a(1) &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} f_a(I_1) \subset (I_1 \cup I_2 \cup I_3) \\ f_a(I_2) \subset I_4 \\ f_a(I_3) \subset I_4 \\ f_a(I_4) \subset (I_1 \cup I_2 \cup I_3) \end{cases},$$

we obtain the graph indicated in Fig. 1(b).

*Corresponding author: desouzapinto@pq.cnpq.br

[1]In the case where $a = 1/3$, the map (1) is reduced to a tent map, for which there are two partitions $0 \leqslant x < 1/2$ and $1/2 \leqslant x \leqslant 1$.

 ©2011 American Physical Society

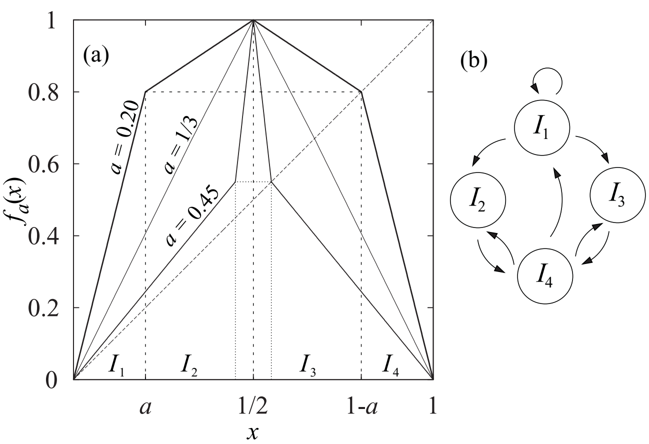

FIG. 1. (a) The bungalow map for three different values of a. The indicated partitions are respective to $a = 0.20$. (b) Possible transitions between partitions of the bungalow map.

Consider, now, the following matrix [26,27]:

$$T \circeq \begin{bmatrix} \eta_1 & \eta_1 & \eta_1 & 0 \\ 0 & 0 & 0 & \eta_2 \\ 0 & 0 & 0 & \eta_3 \\ \eta_4 & \eta_4 & \eta_4 & 0 \end{bmatrix}, \qquad (2)$$

whose eigenvalues are given by $t_{1,2} = (\eta_1 \pm \theta)/2$ and $t_{3,4} = 0$, in which $\theta \equiv \sqrt{\eta_1^2 + 4\eta_4(\eta_2 + \eta_3)}$.

It is straightforward to apprehend that matrix (2), with all $\eta_\alpha = 1$, represents the transfer matrix T_1, associated with the graph in Fig. 1(b), of the map, where the element located in the line ν and column τ of the nth power, $[T_1^n]_{\nu\tau}$ represents the number of different itineraries of size n starting in the partition I_ν and ending in the partition I_τ. Therefore, the topological entropy of map (1) is given by the largest eigenvalue (t_1) logarithm of the matrix T_1 ($h_T = \ln 2$) [27]. Moreover, the invariant density of the map is given by the eigenvector components associated with t_1: $\mathbf{v}_1 = 1/\sqrt{10}[2\ 1\ 1\ 2]^T$ [the component $v_1^{(\alpha)}$ indicates the natural measure of the αth partition].

In matrix (2), the η_α stands for any constant quantity in each interval of linearity I_α and it is multiplicative along a trajectory. Thus, we can use the nth power of the matrix (2) to study the dynamical properties of the map. For example, the diagonal elements of T^n provide the 2^n periodic sequences of size n.

The trace of the matrix is directly related to its eigenvalues by

$$\mathrm{tr}T^n = \sum_\alpha t_\alpha^n.$$

Once we know the eigenvalues of T, we can determine the trace of T^n, whatever the value of n

$$\mathrm{tr}T^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\eta_1^k\theta^{n-k}[1 + (-1)^{n-k}]. \qquad (3)$$

In Eq. (3) each term in the summation is related to a possible symbolic sequence. Thus, if $\eta_\alpha = f'(x)|_{x\in I_\alpha}$, then Eq. (3) gives the stability coefficients spectrum of the nth periodic points of the map (1). As an illustration, we have associated with the intinerary $I_1 I_2 I_4 I_1 I_2 I_4 \cdots \overline{I_1 I_2 I_4}$ a point of period 3. For this case the coefficient of stability is the product $\eta_1\eta_2\eta_4$.

From now on we shall examine the case of two coupled maps. We shall use the following version for the coupling:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{G}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} f_a[x_n + \delta(y_n - x_n)] \\ f_a[y_n + \varepsilon(x_n - y_n)] \end{pmatrix}, \qquad (4)$$

in which δ and ε can assume different (asymmetric coupling) or equal values (symmetric coupling). If any of them vanishes, we obtain a master-slave coupling. At any instance, the dynamics, for synchronization purposes, will depend on their sum $d \equiv \delta + \varepsilon$.

The map $\mathbf{G}$ keeps the unit square invariant when both $0 \leqslant \delta \leqslant 1$ and $0 \leqslant \varepsilon \leqslant 1$ (we will deal only with these intervals). This system has the property to generate dynamics that leave the straight line $x = y$ of the plane invariant and, consequently, the segment $\mathcal{S} = \{(x,y) \in \omega^2 \mid 0 \leqslant x = y \leqslant 1\}$. The latter is often called the *synchronization subspace*.

Since UDV is a local phenomenon, we shall consider the transversal linear stability to the synchronization subspace. So we linearize the system (4) and diagonalize it, in the basis of the Jacobian matrix, in the directions $\mathbf{u}_\parallel = [1\ 1]^T$ and $\mathbf{u}_\perp = [\delta\ \ -\varepsilon]^T$. The quantities associated with the directions $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$ are called longitudinal and transversal, respectively.

By definition, $\mathcal{S}$ is nonhyperbolic if there is at least one periodic point embedded in that subspace whose unstable dimension is different from any other point in $\mathcal{S}$. By construction, all periodic points in the set are longitudinally unstable. Therefore, the phenomenon occurs in this system only in the transversal direction and it is necessary that periodic points transversely stable and unstable coexist with each other in $\mathcal{S}$. To study, in a quantitative way, the unstable dimension variability of the system, we must determine the unstable dimension of *all* periodic points of the map. We must also determine the frequency with which a typical trajectory visits the neighborhood of these points. As in the synchronization manifold the dynamics is hyperbolic and mixing, we know that such frequency can be obtained by the invariant density given by[2] [28]

$$\rho(x) = \frac{1}{\Delta x}\lim_{p\to\infty}\sum_{x\in\mathbb{D}}\frac{1}{|\Lambda_\parallel(x,p)|}, \qquad (5)$$

where $\mathbb{D} = [x, x + \Delta x)$, and the summation extends over all points of period p in $\mathbb{D}$ whose eigenvalues associated with the longitudinal direction[3] are given by $\Lambda_\parallel(x,p)$. Note that expression (3) gives us all possible eigenvalues for all points of period $p = n$. It is possible to calculate, from Eq. (3), the number of periodic points which have the same eigenvalue.

[2]The natural measure, generated by any typical trajectory, of any subset $\mathbb{D} \in \omega$ is given by $\mu(\mathbb{D}) = \int_{\mathbb{D}} \rho(x)\,dx$.

[3]Now we consider the dynamics in the synchronization subspace.

For this purpose, we rewrite θ^{n-k} as follows[4]

$$\theta^{n-k} = \eta_1^{n-k} \sum_{w=0}^{\frac{n-k}{2}} \binom{\frac{n-k}{2}}{w} \left(\frac{2}{\eta_1}\right)^{2w} \sum_{r=0}^{w} \binom{w}{r} \eta_2^r \eta_3^{w-r} \eta_4^w .$$

Replacing in Eq. (3), we obtain

$$\mathrm{tr} T^n = \sum_{k=0}^{n} \binom{n}{k} [1 + (-1)^{n-k}] \sum_{w=0}^{\frac{n-k}{2}} \left(\frac{1}{2}\right)^{n-2w} \times \sum_{r=0}^{w} \binom{\frac{n-k}{2}}{w} \binom{w}{r} \eta_1^{n-2w} \eta_2^r \eta_3^{w-r} \eta_4^w . \quad (6)$$

Equation (6) gives all information required by Eq. (5). Since the system is piecewise linear, the eigenvalues obtained by $\eta_1^{n-2w}\eta_2^r\eta_3^{w-r}\eta_4^w$ can be used to determine in which partition $\mathbb{D}$ these periodic points are contained. Thus, taking the η's as the transversal eigenvalues, we determine the unstable dimension of partition $\mathbb{D}$. On the other hand, taking the η's as the longitudinal eigenvalues, and their coefficients, we determine the measure (i.e., the contribution of this partition to the behavior of typical trajectories in the vicinity of the synchronization manifold). This analysis allows us to quantify the unstable dimension variability.

First, we determine the set of parameters for which the UDV occurs. To calculate the beginning of this phenomenon we evaluate the coefficients of stability of each partition. Therefore, the determination of the parameters a_c and $d_c = \delta + \varepsilon$, which are critical for the beginning and the end of unstable dimension variability, is done by calculating the possible transversal eigenvalues $\eta_1^{n-2w}\eta_2^r\eta_3^{w-r}\eta_4^w$, with

$$\eta_1 = -\eta_4 = \left(\frac{1-a}{a}\right)(1-d),$$

$$\eta_2 = -\eta_3 = \left(\frac{2a}{1-2a}\right)(1-d).$$

Simply, we determine which possible combinations of $\eta_1^{n-2w}\eta_2^r\eta_3^{w-r}\eta_4^w$ result the largest ($\sup|\Lambda_\perp|$) and the lowest ($\inf|\Lambda_\perp|$) eigenvalues, in magnitude, and evaluate the range of the UDV existence as follows:

$$\sup|\Lambda_\perp| = 1 \quad \text{and} \quad \inf|\Lambda_\perp| = 1. \quad (7)$$

Since we are dealing with the magnitude of the eigenvalues, we must consider only two terms. The extremes of the spectrum of eigenvalues are then given by $|\eta_1| = |(\frac{1-a}{a})(1-d)|$ and $|\eta_3\eta_4| = |\frac{2-2a}{1-2a}(1-d)^2|$[5]. From Eq. (7) and solving for d, we have

$$d_c = \begin{cases} 1 \pm \frac{a}{1-a} & \text{for} \quad 0 < a < 1/3, \\ 1 \pm \sqrt{\frac{1-2a}{2-2a}} & \text{for} \quad 1/3 < a < 1/2. \end{cases} \quad (8)$$

[4]Note that the term $[1 + (-1)^{n-k}]$ in Eq. (3) filters only the terms $(n-k)$ which are even.

[5]The itinerary $\overline{I_3 I_4}$ represents the fixed point $x^* = 1 - a$ of the map (1). Exactly in this point the map is nondifferentiable and its invariant density is discontinuous (except for $a = 1/3$). However, the density on the right and left of the point x^* are proportional to $|\eta_3|$ and $|\eta_4|$, respectively. Thus, the itinerary $\overline{I_3 I_4}$ represents a weighted average of the dynamics, in both sides. In the remaining cases, each itinerary of size n is associated with a point of period $p = n$.

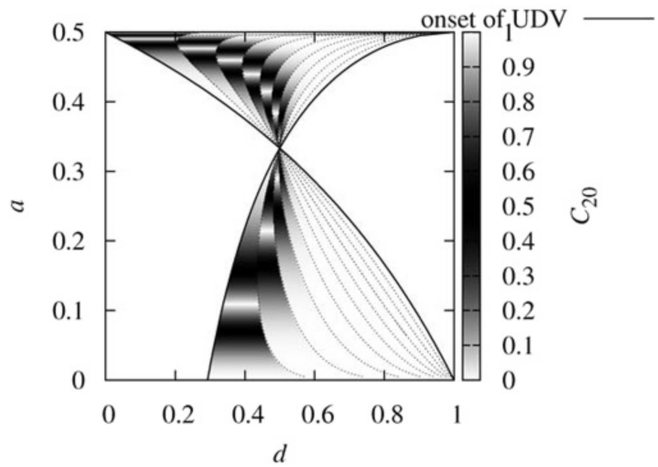

FIG. 2. The UDV intensity, quantified by the contrast measure with $p = 20$, as a function of the parameters a (local dynamics) and d (coupling strength). Solid lines given by Eq. (8), indicate the transition to UDV. Dashed lines denote the transitions between the stabilities of the fixed points of the 20th iteration of the map. This figure is symmetric about $d = 1$.

The dependence of d_c on a is indicated by the solid lines in Fig. 2. Note that for $a = 1/3$ both lines intersect, indicating that there is no UDV in the system for that value of a. This is expected for two tent maps linearly coupled.

The trace T^n shows us the eigenvalues spectrum for an interval of time n, as well as the number of possible eigenvalues. We also know that the diagonal elements of T^n are related to the stability coefficients (eigenvalues) of the n-periodic points.

Reference [3] introduces the quantity

$$C_p \equiv \left| \frac{\mu_2(p) - \mu_1(p)}{\mu_2(p) + \mu_1(p)} \right|, \quad (9)$$

called contrast measure, which quantifies the intensity of UDV. In Eq. (9), the quantities $\mu_i(p)$ read

$$\mu_1(p) = \sum_k \frac{1}{|\Lambda_\parallel(k;p)|} \Theta[1 - |\Lambda_\perp(k;p)|], \quad (10)$$

$$\mu_2(p) = \sum_k \frac{1}{|\Lambda_\parallel(k;p)|} \Theta[|\Lambda_\perp(k;p)| - 1], \quad (11)$$

in which $\Theta(\cdot)$ is the Heaviside function[6]; $\Lambda_\parallel(k;p)$ and $\Lambda_\perp(k;p)$ are the eigenvalue associated with the longitudinal and transversal directions to the synchronization subspace, respectively. These eigenvalues are calculated on the p-periodic point labeled by k. The summation extend over all fixed points of the pth map iteration. Thus, for p large enough, $\mu_{1,2}(p)$ gives the visitation probability of a region with unstable dimension 1 or 2 in the pth map iteration.

Now, using what was described above, we can quantify the UDV from the coefficients in Eq. (6) and Eq. (9). We must observe that the fraction of the positive transversal Lyapunov exponents [29] at p-finite time is *exactly* given by $\mu_2(p)$. This fraction is a metric diagnostic for UDV. If we change $\Theta(\cdot)$ by

[6]We define here central directions ($|\Lambda_\perp| = 1$) as stable ones.

$\ln|\Lambda_{\perp}(k;p)|$ in Eqs. (10) and (11), then $\mu_1(p)+\mu_2(p)$ gives $\langle\lambda_{\perp}(p)\rangle$. So, each term, in summation, gives the contribution for the transversal stability of S of the respective UPO.

Figure 2 shows, in gray scale, the intensity of UDV, quantified by the contrast measure, in the space parameter. Observing the figure, we notice a large region, limited by the solid lines, in which the system is nonhyperbolic ($C_p \neq 1$). There is also a large region in which UDV is weak ($C_{20} \lessapprox 1$). For these values of C_20 the set of periodic orbits responsible for UDV has positive measure, but a very small one. Thus, a numerical diagnostic of nonhyperbolity, as the fraction of the positive finite-time Lyapunov exponent, typically cannot identify such regions.

In conclusion, we have seen that the synchronized subspace of two coupled bungalow maps presents four intervals of linearity, which define four Markov partitions of the phase space. Since UDV does not occur in the longitudinal direction of this subspace, we are able to study analytically the symbolic and interval dynamics in $\mathcal{S}$. Pursuant to this study we found the stability coefficients of periodic points of the dynamics in the subspace synchronization, which, in turn, allowed us to write an exact expression for the contrast measure. Thus, we establish analytical solutions showing the onset of UDV, as well as the transitions between the stability of periodic points in parameter space. We can use this result to identify regions in parameter space as long as the solutions remain valid.

This work has only been able to touch on a simple dynamical system. However, the preliminary study reported here has highlighted the need to explore the possibilities of finding analytical solutions to the UDV problem. As this issue involves the validity of numerical solutions is important to have exact solutions for models that are studied. Clearly, further studies are needed to understand the UDV for systems with higher dimensions and arbitrary elements. To carry on this research we intend to study the UDV in a coupled map lattice whose couplings changes over time.

This work has been made possible thanks to the partial financial support from the following Brazilian research agencies CNPq, CAPES, and Fundação Araucária.

[1] E. Kostelich, I. Kan, C. Grebogi, E. Ott, and J. Yorke, Physica D **109**, 81 (1997).
[2] Y. C. Lai and C. Grebogi, Phys. Rev. Lett. **77**, 5047 (1996).
[3] Y. C. Lai, Phys. Rev. E **59**, R3807 (1999).
[4] R. F. Pereira, S. Camargo, S. E. de S. Pinto, S. R. Lopes, and R. L. Viana, Phys. Rev. E **78**, 056214 (2008).
[5] T. D. Sauer, Phys. Rev. E **65**, 036220 (2002).
[6] T. Sauer, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **79**, 59 (1997).
[7] R. L. Viana, S. E. de S. Pinto, and C. Grebogi, Phys. Rev. E **66**, 046213 (2002).
[8] P. Schmelcher and F. K. Diakonos, Phys. Rev. Lett. **78**, 4733 (1997).
[9] R. L. Davidchack and Y. C. Lai, Phys. Rev. E **60**, 6172 (1999).
[10] Y. Nagai and Y. C. Lai, Phys. Rev. E **55**, R1251 (1997).
[11] Y. Nagai and Y. C. Lai, Phys. Rev. E **56**, 4031 (1997).
[12] R. F. Pereira, S. E. de S. Pinto, R. L. Viana, S. R. Lopes, and C. Grebogi, Chaos **17**, 023131 (2007).
[13] R. Davidchack and Y. Lai, Phys. Lett. A **270**, 308 (2000).
[14] Y.-C. Lai and C. Grebogi, Int. J. Bifurcation Chaos **10**, 683 (2000).
[15] S. Dawson, C. Grebogi, T. Sauer, and J. A. Yorke, Phys. Rev. Lett. **73**, 1927 (1994).
[16] G. T. Kubo, R. L. Viana, S. R. Lopes, and C. Grebogi, Phys. Lett. A **372**, 5569 (2008).
[17] J. D. Szezech Jr., S. R. Lopes, R. L. Viana, and I. L. Caldas, Physica D **238**, 516 (2009).
[18] J. D. Szezech Jr., S. R. Lopes, I. L. Caldas, and R. L. Viana, Physica A **390**, 365 (2010).
[19] N. Nirmal Thyagu and N. Gupte, Phys. Rev. E **79**, 066203 (2009).
[20] Y. C. Lai and C. Grebogi, Phys. Rev. Lett. **82**, 4803 (1999).
[21] R. Viana, C. Grebogi, S. Pinto, S. Lopes, A. Batista, and J. Kurths, Phys. Rev. E **68**, 067204 (2003).
[22] R. L. Viana, C. Grebogi, S. E. de S. Pinto, S. R. Lopes, A. M. Batista, and J. Kurths, Physica D **206**, 94 (2005).
[23] R. F. Pereira, S. E. de. S. Pinto, and S. R. Lopes, Physica A **389**, 5279 (2010).
[24] W. Steeb, M. van Wyk, and R. Stoop, Int. J. Theor. Phys. **37**, 2653 (1998).
[25] M. C. Verges, R. F. Pereira, S. R. Lopes, R. L. Viana, and T. Kapitaniak, Physica A **388**, 2515 (2009).
[26] R. Stoop and W. H. Steeb, Phys. Rev. E **55**, 7763 (1997).
[27] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen, 2010).
[28] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).
[29] R. L. Viana, S. E. de S. Pinto, J. R. Barbosa, and C. Grebogi, Int. J. Bifurcation Chaos **13**, 3235 (2003).