In this Letter we focus on a kind of non-hyperbolic behavior, for which there is no continuous splitting between stable and unstable subspaces along the chaotic invariant set. This is due to the existence of periodic orbits, embedded in the chaotic set, with a different number of unstable directions, or the unstable dimension variability (UDV) [5]. A consequence of UDV is the absence of shadowing trajectories long enough to warrant a priori numerical computations using chaotic pseudo-trajectories [6]. Even though for some systems this seems not to be an essential difficulty, since the physically interesting quantities are of statistical nature, like averages and variances [7], even statistical quantities may be plagued with an uncontrollable amplification of the noise inherent to computer-generated trajectories [8].

A fingerprint of UDV is that the finite-time Lyapunov exponent nearest to zero fluctuates about zero due to visits of the trajectory to regions of the attractor with a varying number of stable and unstable directions [9]. Other characterization of UDV is based on the behavior of the natural measure of the chaotic set in the vicinity of periodic orbits with different unstable dimension: the scaling of the probability distribution of the attractor near each periodic point agrees with an analytical estimate for the pointwise dimension [10]. The direct verification of UDV, however, is still restricted to a few dynamical systems, and sometimes only to certain intervals of their parameters [11].

The onset of UDV has been described, up to now, by two mechanisms, both requiring the dynamical systems to possess an invariant subspace with a chaotic set, for which there is a tunable bifurcation parameter. The first, known as bubbling transition or riddling bifurcation, describes the onset of UDV at the parameter value in which a low-period periodic orbit embedded in the chaotic set loses transversal stability [12–15]. The second mechanism, closely related to the first one, involves an unstable periodic orbit whose period goes to infinity as we approach the onset of UDV [16]. In this Letter we report a third mechanism, qualitatively different from the previous ones, since it does not require the existence of an invariant subspace with a chaotic attractor in the phase space, and whereby the onset of UDV is triggered by an interior crisis, or a collision between a chaotic attractor and an unstable periodic orbit. This crisis-induced UDV is herewith exemplified by a mechanical system, the kicked double rotor [11].

We work with invertible discrete-time maps, but bearing in mind that continuous time flows may also be described by them using Poincaré sections. Let \( x_\alpha = F(x_\alpha, \alpha) \) be such an \( N \)-dimensional system, with the bifurcation parameter \( \alpha > 0 \). We assume that there is an interval of \( \alpha \)-values for which this map has a chaotic attractor \( A \) of dimension less than \( N \). There is an in-
finite number of unstable periodic orbits embedded in $\mathcal{A}$, with $N_5$ ($N_U$) stable (unstable) directions, corresponding to eigenvalues of the tangent map $\mathbf{D}f(x)$ with moduli smaller (greater) than unity, such that $N = N_U + N_5$. As the bifurcation parameter $\alpha$ crosses the critical value $\alpha_c$ from below, we assume that the unstable dimension increases ($N_U \rightarrow N_U + N'$ and $N_5 \rightarrow N_5 - N'$, where $N'$ is an integer), which marks the onset of UDV.

In the first scenario, the bubbling-type transition, the attractor $\mathcal{A}$ belongs to an invariant subspace of the $N$-dimensional phase space. Moreover, it is assumed that, when $\alpha = \alpha_c$, there is a bifurcation for which a low-period periodic orbit embedded in $\mathcal{A}$ becomes transversely unstable as well as all its pre-images. As a simple example, though representative of a large class of dynamical systems undergoing this type of transition to UDV, let us consider the following ($N = 2$)-dimensional map:

$$x_{n+1} = f(x_n),$$

$$y_{n+1} = g(x_n, \alpha)h(y_n),$$

where the $x$-dynamics is supposed to have a chaotic attractor $\mathcal{A}$, whose existence does not depend on the bifurcation parameter $\alpha > 0$. We also assume the existence of a symmetry in the system so as to warrant the condition $h(-y) = -h(y)$ and, as a consequence, that $y = 0$ is an invariant subspace where the attractor $\mathcal{A}$ lies.

We also assume that there exists a critical value $\alpha_c$ such that, for $\alpha < \alpha_c$ all periodic orbits embedded in $\mathcal{A}$ are transversely stable (their eigenvalues along the $y$-direction have moduli less than unity), hence they have unstable dimension $N_U = 1$ (along the $x$-direction of the attractor) [Fig. 1(a)]. The bubbling transition occurs through a bifurcation at $\alpha = \alpha_c$, for which a periodic orbit loses transversal stability since the eigenvalue along the $y$-direction has crossed the unit circle [13,14]. Hence, for $\alpha > \alpha_c$ this orbit, as well as its infinite pre-images, has acquired $N_U = 2$. Since this set of transversely unstable repellers is dense and coexists in the attractor $\mathcal{A}$ (for $\alpha > \alpha_c$) with the set of transversely stable saddles, we have UDV in this case. We stress that, just after the bubbling transition the attractor has to be understood in the Milnor sense, i.e., its basin of attraction does not need to include the whole neighborhood of the attractor [17].

By contrast, for the crisis-induced scenario proposed in this Letter, there is no such bifurcation point but there is a heteroclinic crossing: for $\alpha < \alpha_c$ a chaotic attractor $\mathcal{A}$ has orbits with unstable dimension $N_U$, and $\mathcal{A}$ coexists with a periodic orbit having $N_U + N'$ unstable directions, where $N'$ is an integer. When $\alpha = \alpha_c$ this unstable periodic orbit collides with the pre-critical attractor $\mathcal{A}$ and, as a result of this interior crisis, $\mathcal{A}$ may suffer three types of changes [18]: (i) it can suddenly be destroyed into a non-attracting chaotic saddle; (ii) it can suffer a sudden widening with respect to the pre-critical attractor; or (iii) two or more pre-critical chaotic attractors can merge forming a single and large post-critical chaotic attractor. Either one of these changes can lead to UDV, since in both cases the post-critical attractor $\mathcal{A}$ contains orbits with both unstable dimensions $N_U$ and $N_U + N'$. In terms of the two-dimensional map given as a paradigmatic example, the pre-critical scenario consists of a chaotic attractor $\mathcal{A}$ (though, in general, it does not have necessarily to belong to an invariant subspace, like in this example) wherein all periodic orbits are transversely stable ($N_U = 1$). Moreover, there is a repeller ($N_U = 2$) outside the chaotic attractor $\mathcal{A}$, which is a transversely unstable periodic orbit of the $y$-dynamics [Fig. 1(b)]. If it is a fixed point, for example, its coordinates are given by $y^* = g(x^*, \alpha)h(y^*)$, for $\alpha < \alpha_c$; whereas we assume that the orbit collides with the attractor at $x = x^*$ and thus has $y^* = 0$ for $\alpha > \alpha_c$. For $\alpha > \alpha_c$ we can assume, without loss of generality, that the post-critical attractor $\mathcal{A}$ does not disappear and contains both those transversely stable orbits as well as the transversely unstable repeller and all its pre-images which, because of ergodicity, are dense on $\mathcal{A}$. As a consequence, in the post-critical attractor there coexist orbits with different number of unstable directions.

While the bubbling-type scenario for explaining the onset of UDV requires the chaotic attractor to belong to an invariant subspace of the system, the crisis-induced UDV mechanism can be observed in systems where the chaotic attractor lies outside such an invariant subspace (it may not even exist at all), thus enlarging considerably the class of dynamical systems describable by this scenario. Furthermore, interior crises are quite common in dynamical systems of physical interest, so the crisis-induced scenario for UDV would be often observed [18]. The increase of unstable dimension caused by the onset of UDV is believed to occur often in high-dimensional dynamical systems, like chains of coupled chaotic continuous-time oscillators or discrete-time maps [19,20].

As a physically interesting example of crisis-induced UDV, whose dimensionality is not too large to hinder a detailed de-
variables are plotted versus the kick strength $f$. Double rotor map is depicted in Fig. 2, where the four phase space tractor until it escapes into the added post-critical region made in the phase space region corresponding to the pre-critical at-period-6 unstable orbit, as shown in Fig. 3.

After the interior crisis the chaotic orbit stays for a long time in the phase space region corresponding to the pre-critical attractor until it escapes into the added post-critical region made available by the chaotic saddle related to the pre-critical period-6 orbit. These transient excursions through the new attractor regions have a characteristic time scale $\tau$, whose average values are plotted in Fig. 4 against the parameter difference $f_0 - f_{0CR}$ just after the interior crisis has occurred. We verified the expected power-law scaling

$$\langle \tau \rangle \sim (f_0 - f_{0CR})^{-\gamma},$$

where we estimated $\gamma = 1.358$. For $f_0 \lesssim f_{0CR}$ the characteristic time is infinite, since an orbit remains forever on the pre-critical chaotic attractor, provided it starts in a point belonging to its basin of attraction. If, however, we add noise to the system with strength $\sigma$, it is possible that an orbit starting from the pre-critical attractor will behave like the orbit of the post-critical attractor even for $f_0 \lesssim f_{0CR}$, thereby yielding a transient response. In Ref. [22], it was shown that the characteristic time of such a noisy system scales as

$$\langle \tau \rangle \sim \sigma^{-\gamma} \left[ \frac{(f_0 - f_{0CR})^{\gamma}}{\sigma} \right],$$

where $g(.)$ is a non-universal function depending on the system and the probability distribution function of the noise, and $\gamma$ is the same critical exponent of the system without noise.
Let us consider the pre-critical chaotic attractor $A$, which exists in the parameter interval $f_{0CH} < f_0 < f_{0CR}$, in order to understand the onset of UDV for this system. For this to occur it is sufficient to show the coexistence of periodic orbits that have different number of unstable directions as compared to those embedded in the chaotic attractor, and that one of those unstable periodic orbits does collide with the chaotic attractor and it will be embedded in it after the collision. According to [Fig. 5(a)], there are saddle orbits embedded in $A$ with unstable dimension $N_u = 1$ and stable dimension $N_s = 3$. There are also, for this same interval of $f_0$, unstable orbits outside $A$ with unstable dimension $N_u = 2$ (and $N_s = 2$). For example, the unstable subspace of the fixed point $\mathbf{0}$ is two-dimensional for all values of $f_0$ [Fig. 5(a)], whereas the fixed point $\mathbf{Q} = (0, \pi, 0, 0)^T$ acquires $N_u = 2$ after $f_0 = 4.5114$ and remains so [Fig. 5(b)]. Since both of them remains outside $A$ before the crisis at $f_{0CR}$, they are engulfed by the post-critical attractor and become part of it, what characterizes UDV since the post-critical attractor has embedded orbits with both $N_u = 1$ and $N_u = 2$.

A sufficient condition for this to scenario to take place is that one of those fixed points, say $\mathbf{0}$, belongs to the post-critical attractor but not to the pre-critical one. A direct way to verify this condition would be to show that the closure of the unstable manifold stemming from $\mathbf{0}$ is the post-critical attractor. In fact, we observe that a numerical approximation for one of the unstable manifold branches $W^u(\mathbf{0})$ stemming from $\mathbf{0}$ in the immediately post-critical regime seems to asymptote to the former pre-critical attractor. In Fig. 6(a) we plot the pre-critical chaotic attractor, using length-100 orbits from a large number (1000) of initial conditions and discarding the first 79900 transient iterations. The immediate post-critical situation is depicted by Fig. 6(b), which was obtained by considering a disk of small radius ($\sim 10^{-6}$) centered at $\mathbf{0}$ and containing a large number (15000) of initial conditions. We computed numerically the first 25 forward images of this disk and taken their union as a numerical approximation for $W^u(\mathbf{0})$. We guess, from Fig. 6(b), that these images asymptote to the former pre-critical attractor such that the latter is contained in the closure of $W^u(\mathbf{0})$. However, (the) this is quite difficult to show in such a high-dimensional phase space since one is dealing with two-dimensional projections.

Another way to argue that the fixed point $\mathbf{0}$ belongs to the post-critical attractor is to investigate the behavior of the natural measure of the attractor in the vicinity of the fixed point. Accordingly we took initial conditions very close to the fixed point ($10^{-7}$ apart from $\mathbf{0}$) and iterate the map a large number of times ($\sim 10^9$), counting how often the resulting trajectory entered a ball centered at $\mathbf{0}$ with radius $\epsilon$. The ratio between this number and the total number of trajectory points is a numerical approximation of the natural measure of the post-critical attractor in that vicinity. The radius $\epsilon$ cannot be made too small, since the fixed point $\mathbf{0}$ has two unstable directions and thus is scarcely visited by trajectories in the post-critical attractor. On the other hand, we can use rather large values for $\epsilon$ (of the order of 0.1) since, from Fig. 2, a ball centered at $\mathbf{0}$ and with $\epsilon$-values as large as the unity is not likely to cross other relevant fixed points in the post-critical attractor. Our results are summarized in Fig. 7, where the natural measure in this vicinity for two values of $\epsilon$ is plotted against the distance between the bifurcation parameter and its critical value. It is clearly seen that the natural measure is nonzero in a consistent way after the crisis has occurred, and we take this as a confirmation of the fixed point $\mathbf{0}$ belonging to the post-critical attractor.

Other low-period periodic orbits not belonging to the pre-critical attractor and with $N_u \neq 1$ can be engulfed by the post-critical attractor when it is enlarged after the interior crisis so as to cover the entire $(0, 2\pi)$ domain of the variables $(x_1, x_2)$ and a large domain of the other ones [Fig. 2]. Actually the two fixed points $\mathbf{0}$ and $\mathbf{Q}$ seem to be connected, in the post-critical regime, by heteroclinic orbits, as illustrated in the section $y_1 = y_2 = 0$ shown in Fig. 8, which was obtained from a large (200 000) number of randomly chosen initial conditions in the $(x_1, x_2)$ plane, each of them
generating a very long orbit (of $2 \times 10^6$ iterates, the first 6000 transient ones being discarded). The fixed points $\mathbf{0}$ and $\mathbf{Q}$ are also connected with $\mathbf{P}$ by a wiggling surface whose section even exhibits the characteristic folding of heteroclinic higher-dimensional saddles. Hence, regardless of taking $\mathbf{0}$ or $\mathbf{Q}$ as the colliding orbits with different unstable dimensions, we conclude that the interior crisis point $f_{\text{OCR}}$ marks the onset of UDV for the kicked double rotor.

Besides the direct methods we have just described, we have also verified the onset of UDV in this particular dynamical system by using an indirect characterization, which is the statistical behavior of the second finite-time Lyapunov exponent (because it is the closest to zero in the Lyapunov spectrum). A fingerprint of the occurrence of UDV in a dynamical system is the fluctuating behavior about zero of the finite-time exponent closest to zero [9]. The $k$th time-$n$ Lyapunov exponent for the point $\mathbf{x}_0$ is defined as [23]

$$\lambda_k(\mathbf{x}_0, n) = \frac{1}{n} \ln |\mathbf{DF}(\mathbf{x}_0) \cdot \mathbf{v}_k| \quad (i = 1, \ldots, 4),$$

where $\mathbf{v}_k$ is the singular vector corresponding to $\xi_k(\mathbf{x}_0, n)$, the singular value of the Jacobian matrix of $\mathbf{DF}(\mathbf{x}_0)$. Although the time-$n$ exponent $\lambda_k(\mathbf{x}_0, n)$ generally takes on a different value, depending on the initial condition $\mathbf{x}_0$, the infinite-time limit of $\lambda_k(\mathbf{x}_0, n)$ has the same value for almost all $\mathbf{x}_0$ with respect to the natural ergodic measure of the invariant set.

For the kicked double rotor map the finite-time exponent closest to zero is $\lambda_2(\mathbf{x}_0, n)$ and, as long as its chaotic attractor displays UDV, there will be length-$n$ sections of a typical trajectory for which $\lambda_2(\mathbf{x}_0, n)$ is either positive or negative. This is equivalent to say that the probability distribution function $P(\lambda_2(\mathbf{x}_0, n))$ has positive values, i.e., a nonzero fraction of positive values for $\lambda_2(\mathbf{x}_0, n)$:

$$\phi(\lambda_2(n)) = \frac{\int_{-\infty}^{+\infty} P(\lambda_2(\mathbf{x}_0, n)) d\lambda_2}{\int_{-\infty}^{+\infty} P(\lambda_2(\mathbf{x}_0, n)) d\lambda_2}$$

(9)

We obtained numerical approximations for this probability distribution using a large number of length-$n$ ($n = 15$) chaotic trajectories starting from randomly chosen initial conditions on the attractor. The onset of UDV can be related to the parameter value at which the fraction $\phi(n)$ crosses zero. According to Fig. 9(a), within the numerical accuracy, this occurs at the interior crisis point $f_{\text{OCR}}$. It is worth considering the behavior of this fraction also in the vicinity of the $f_0$ value where UDV has maximum strength [Fig. 9(b)]. In this vicinity the positive fraction increases monotonically and at $f_{\text{0H}} = 7.9632$ it reaches 1/2, so as to indicate that the positive and negative contributions of $\lambda_2(n)$ are the same and $\phi(\lambda_2(n)) = 0$ [24]. An important consequence is that the infinite-time exponent, in this case, is zero at $f_{\text{0H}}$, and, for $f_0 \geq f_{\text{0H}}$, the system becomes hyper-chaotic. The existence of periodic orbits with different unstable dimensions embedded in the hyper-chaotic attractor (around $f_0 = 8.0$) was the first evidence of UDV in a physical system ever given [11].

This transition to hyper-chaos also marks the point at which the effect of UDV is of maximum strength, since the number of trajectory pieces which is contracting (on average) along the direction $k = 2$ does counterbalance that of the expanding pieces. Hence the shadowing time $T_S$, or the time interval during which a noisy pseudo-trajectory is shadowed by a “true” chaotic one, may be rather small to warrant the validity of long numerically generated chaotic trajectories [46]. In fact, using a theoretical model based on a biased random walk with a reflecting barrier, the average shadowing time was shown to obey a power-law scaling with respect to the one-step error $\delta$ as [6].
\[ \langle T_S \rangle \sim \delta^{-h}, \quad h \equiv \frac{2|m|}{\sigma^2} \tag{10} \]

in terms of the average \( m = \langle \lambda_2(n) \rangle \) and variance \( \sigma^2 \) of the probability distribution function \( P(\lambda_2(n)) \). The one-step error \( \delta \) can be estimated as \( 10^{-10} \), which is the numerical precision obtained with the help of a double-precision arithmetics in 32-bit computers. When \( \langle \lambda_2(n) \rangle = 0 \), at \( f_{0h} \approx 7.9632 \), \( h \) vanishes and the average shadowing time takes on its minimum possible value. We remark, however, that such a stochastic approximation is valid only at the vicinity of the point where UDV is most severe, i.e., at \( h \approx 0 \), since the behavior of the fluctuating finite-time exponent is more closely modelled by a biased random walk with reflecting barrier.

In conclusion, we have presented a novel scenario for the onset of unstable dimension variability in dynamical systems characterized by an interior crisis, or the collision between a chaotic attractor and an unstable periodic orbit. The essential requirement is that there exists an unstable periodic orbit in the post-critical attractor with a different number of unstable directions from the periodic orbit with whom it collides. After the collision both orbits coexist in the same post-critical attractor. We argued that this scenario for the onset of unstable dimension variability is more general than the bubbling-type one, which requires the existence of an invariant subspace to which a chaotic attractor belongs. On the other hand, in the proposed crisis-induced scenario, there is no need of such invariant subspace.

The range of possible applications of our proposed mechanism is quite wide. One of the reasons is that most dynamical systems of physical interest are non-hyperbolic and thus present, in their bifurcation diagrams, chaotic regions interspersed with periodic windows. As a system parameter is varied, such a periodic window typically starts from a saddle-node bifurcation which creates a pair of stable/unstable periodic orbits which bifurcate (through a period-doubling cascade), generating a many-band chaotic attractor. The periodic window ends abruptly when this chaotic attractor collides with the unstable periodic orbit created at the beginning of the window, through an interior crisis of the same type as described in this Letter.

Acknowledgement

This work was partially supported by CNPq, CAPES, and Fundação Araucária (Brazilian Government Agencies).

References