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Lyapunov spectrum of a lattice of chaotic systems with local and non-local couplings

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Abstract

We consider a one-dimensional chaotic piecewise linear map lattice with periodic boundary conditions and two types of interactions: (i) local couplings between nearest and next-to-the-nearest neighbors; and (ii) non-local couplings randomly chosen along the lattice according to a specified probability. The chaoticity of the lattice is described by means of its Lyapunov spectrum, which furnishes also information about the system global attractor in a high-dimensional phase space. We study in particular the dependence of this spectrum with the coupling parameters, as well as make comparisons with limiting cases, for which the Lyapunov spectrum is known.
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1. Introduction

Many lattices of interest in physical, biological and social contexts exhibits the *small-world property*: with a relatively small number of connections one can reach any site of the lattice [1]. In social networks this has been related to the so-called *six degree-of-separation* problem [2]. The propagation of diseases [3] and the structure of large computer programs [4] have also been reported to exhibit evidences of the small-world property. In the neuro-science literature, recent works deal with the small-world property in the connection architecture of biological neural networks, for example in the cat brain cortex [5].

The small-world property has been implemented in a practical way in spatially extended systems from the seminal work of Watts and Strogatz [6], who have introduced regular coupled lattice systems (with nearest neighbors) where a small number of connections are rewired to become non-local shortcuts according to a specified probability. The Watts–Strogatz lattice exhibits the small-world property for a certain range of the probability of non-local shortcuts.

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In this case, the average number of degrees-of-separation between any two sites of the lattice is practically as small as in random networks [7]. On the other hand, the Watts–Strogatz lattice also presents a large amount of clustering, meaning that there are comparatively many clusters of sites which are also connected to each other. While this property is typical for regular lattices, it is almost absent in purely random ones.

Since the non-local shortcuts are produced by rewiring some of the local connections, there is a nonzero probability of producing sub-lattices disconnected or weakly connected to the rest of the lattice. In order to overcome this problem, Newman and Watts have proposed a variant of the Watts–Strogatz lattice in which the shortcuts are not produced by rewiring but are instead added to the lattice according to a specified probability [8]. In spite of this difference, Newman–Watts lattices have similar properties as Watts–Strogatz lattices, and have been shown to be likewise useful in applications involving small-world phenomena [9].

In this work we consider an implementation of the Newman–Watts lattices to chains of piecewise linear maps which, when uncoupled, exhibit a strongly chaotic dynamics. The presence of non-local shortcuts in the lattice has been long known to induce and enhance synchronization properties of the lattice [10]. On the other hand, there are still some open questions about the role of the non-local connections on the spatio-temporal dynamics generated by small-world lattices. In particular, how the chaoticity of the lattice as well as the dimension of the corresponding attractor in the high-dimensional phase space are affected by an increase of the shortcut probability. We investigate numerically the Lyapunov spectrum of a coupled map lattice with regular and random couplings. The results allow us to make a comparison between a related class of coupling, namely a prescription for which the interaction strength decreases with the lattice distance as a power-law [11].

This paper is organized as follows: in Section 2 we introduce the coupled map lattice model with regular and random connections and its Lyapunov spectrum, as well as the quantities we obtain from it to analyze the spatio-temporal patterns observed in numerical simulations. Section 3 deals with map lattices with a power-law coupling, for which the Lyapunov spectrum is known in some detail for some particular cases. Section 4 contains our conclusions.

2. Map lattice with regular and random couplings

Let us consider a one-dimensional lattice of N maps, each of them with its state variable at discrete time n: $x_n^{(i)}$, where $i = 1, 2, \ldots, N$. A model presenting regular as well random couplings is [12]

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \frac{\epsilon}{4+M}\left[f(x_n^{(i-1)}) + f(x_n^{(i-2)}) + f(x_n^{(i+1)}) + f(x_n^{(i+2)}) + \sum_{j=1}^{M} f(x_n^{(j)}) I_{ij} \right], \tag{1}$$

where $\epsilon > 0$ is the coupling strength, and the local dynamics at each site is represented by a piecewise linear map $x \mapsto f(x) = \beta x \pmod 1$. For $\beta > 1$ the uncoupled maps display strong chaos, with Lyapunov exponent $\lambda_U = \ln \beta$ (from now on we choose $\beta = 3$).

Our coupling scheme has two contributions: (i) a regular one, from the nearest and second-nearest neighbors of a given site; and (ii) a random term represented by the connectivity matrix elements I_{ij}. The elements of this matrix are 1s and 0s, provided the sites $x^{(i)}$ and $x^{(j)}$ are or not coupled, respectively. In principle, a straightforward implementation of the Newman–Watts lattice would be achieved by randomly choosing the nonzero entries of the connectivity matrix, what in general yields a different number of connections *per site*. However, in doing so we exclude, in general, the possibility of having a completely synchronized state in the lattice [13]

$$x_n^{(1)} = x_n^{(2)} = \cdots x_n^{(N)} \equiv \chi_n. \tag{2}$$

Such a completely synchronized state must be an allowed solution of the coupled map lattice given by Eq. (1), what imposes a constraint on the form of the connectivity matrix. In order to have a synchronized state we assume a fixed number M of randomly chosen shortcuts for each site, chosen with a uniform probability $p = M/N$. Hence, in each row of the connectivity matrix I_{ij} there are M entries equal to unity, the $N - M$ others being padded with zeroes, such that the following condition holds:

$$\sum_{j=1}^{M} I_{ij} = M = Np \quad (i = 1, 2, \ldots, N), \tag{3}$$

which means that, for an arbitrary value $0 < p < 1$, each site has out of M non-local shortcuts with another sites, besides its four local connections with its nearest and next-to-nearest neighbors. The distribution of the number k of connections *per* lattice site, $P(k)$, is thus singular with one delta peak in $k = 4 + M$. By way of contrast, in a scale-free network this distribution would go with k as a power-law (k^{β}), with $\beta > 0$ [14].

Supposing that we are in a completely synchronized state like (2) there results, from Eq. (1), that

$$\chi_{n+1} = (1-\epsilon)f(\chi_n) + \frac{\epsilon}{4+M}\left[4f(\chi_n) + f(\chi_n)\sum_{j=1}^{M} I_{ij}\right] = f(\chi_n), \tag{4}$$

which leads to the condition (3).

A lattice with N coupled one-dimensional maps is a N-dimensional dynamical system, and the corresponding Lyapunov spectrum is formed by N exponents, one for each independent eigendirection in the tangent space: $\lambda_1 = \lambda_{\max} > \lambda_2 > \cdots > \lambda_N$. The maximal Lyapunov exponent $\lambda_{\max}$ represents the exponential rate at which an arbitrarily small displacement is amplified with time, so it suffices that $\lambda_{\max} > 0$ to, technically speaking, ensure that the lattice dynamics is chaotic. The sum of all exponents $\lambda_1 + \lambda_2 + \cdots + \lambda_N$ yields the rate of growth of a volume element in the N-dimensional phase space.

In a system of coupled chaotic maps, it may well happen that many exponents are positive, hence a quantity of interest is the lattice-averaged value of the positive Lyapunov exponents [15]

$$h = \langle \lambda_i \rangle_{i,\lambda_i>0} = \frac{1}{N}\sum_{i=1}^{\lambda_i>0} \lambda_i. \tag{5}$$

According to Pesin's theorem, if the local dynamics is generated by a diffeomorphism with an invariant ergodic measure absolutely continuous with respect to the Lebesgue measure, this quantity is equal to the density of Kolmogorov–Sinai (KS) entropy [16]. It is also an upper bound for the KS-entropy if the local dynamics is a C^1-map preserving an ergodic measure [17]. The equality between h and the density of KS-entropy is generally valid for systems—like Axiom-A systems—having a Sinai–Ruelle–Bowen measure [18]. Such measures have been found for certain coupled map lattices [19].

In the following we shall consider lattices with $N = 201$ sites with periodic boundary conditions, $x_n^{(i)} = x_n^{(i\pm N)}$, and initial conditions $(x_0^{(i)})$ randomly chosen within the interval $[0,1]$ according to a uniform distribution. The dependence of the KS-entropy density h with the probability p of non-local shortcuts and the coupling strength ϵ is shown in Fig. 1, where we let the system evolve a very large number of time iterations, such that we expect transients to have died out.

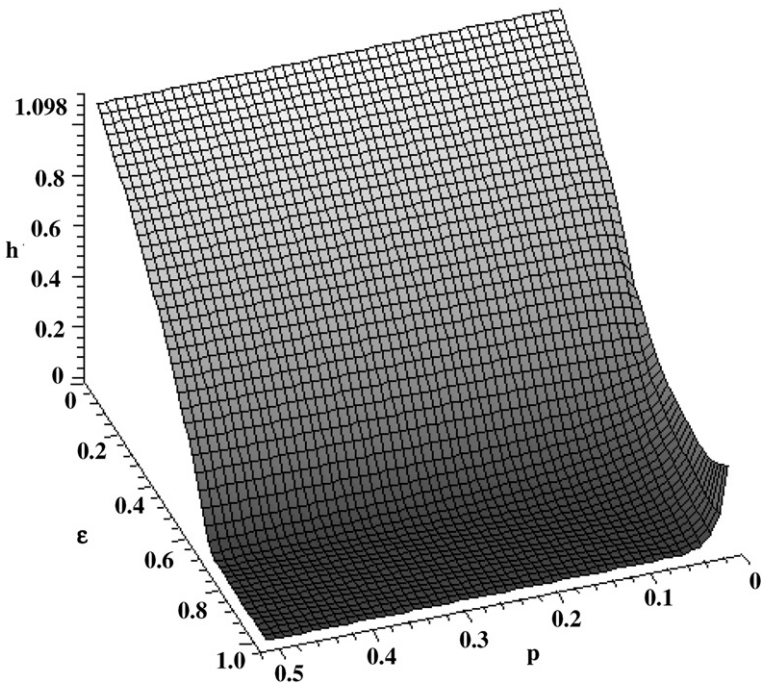

Fig. 1. Density of KS-entropy for a lattice with $N = 201$ maps, $\beta = 3$, and regular and random couplings, as a function of the probability of shortcuts p and the coupling strength ϵ.

If the maps are uncoupled the entropy density h would evidently reduce to the single Lyapunov exponent $\lambda_U = \ln 3 \approx 1.098$. As the coupling strength is increased, Fig. 1 indicates a monotonic decrease of the lattice entropy. The rate at which this decrease occurs is quite independent of the shortcut probability as long as ϵ is less than a value around 0.4. Otherwise the dependence of h with ϵ turns to be quite different for small p. For no shortcuts at all ($p = 0$), we recover a purely regular lattice with local couplings, for which the entropy is known to reach a minimum at $\epsilon \approx 0.8$ and, after that, it starts to increase for higher ϵ. It has been long known that such a locally coupled lattice with a very strong coupling increases its entropy [20].

The addition of randomly chosen shortcuts is shown in Fig. 1 to reduce this effect. Not only the decreasing of h becomes monotonic with higher p, but its value decreases rapidly to a value which, as we will see, is small but nonzero. Moreover, this transition to a small value of h becomes sharper with increasing p. In the limit case of $p = 1$, for which all lattice sites have N non-local connections, the transition occurs for a critical value of $\epsilon \approx 0.7$.

The fact that, for a shortcut probability large enough, the entropy assumes a small yet nonzero value gives us a clue about the system dynamics in phase space. A useful quantity is, in this case, the Lyapunov dimension D. Let p be the largest integer for which $\sum_{j=1}^{p} \lambda_j \geqslant 0$. Then D is defined by one of the following relations [21]:

$$D = \begin{cases} 0, & \text{if there is no such } p, \\ p + \frac{1}{|\lambda_{p+1}|} \sum_{j=1}^{p} \lambda_j, & \text{if } p < N, \\ N, & \text{if } p = N. \end{cases} \tag{6}$$

The Lyapunov dimension furnishes an estimate of the information dimension of the system attractor embedded in the N-dimensional phase space of the coupled map lattice, and which is the dimension associated to the most representative part of its natural measure [22]. Hence, D can give us a hint on the complexity of the phase space for such a spatially extended system. Our results are depicted in Fig. 2 for a lattice with $N = 201$ coupled maps. When the maps are uncoupled we have a maximum value for the Lyapunov dimension: $D_{\max} = N$. When ϵ builds up and the maps become progressively more strongly coupled, we expect the Lyapunov dimension to decrease.

Actually the properties of the Lyapunov dimension are quite similar to the KS-entropy density. The difference between these quantities is that the summation giving the integer p in (6) considers the Lyapunov exponents for which the summation itself is non-negative. The Pesin's formula for KS-entropy (5), however, considers the summation of the

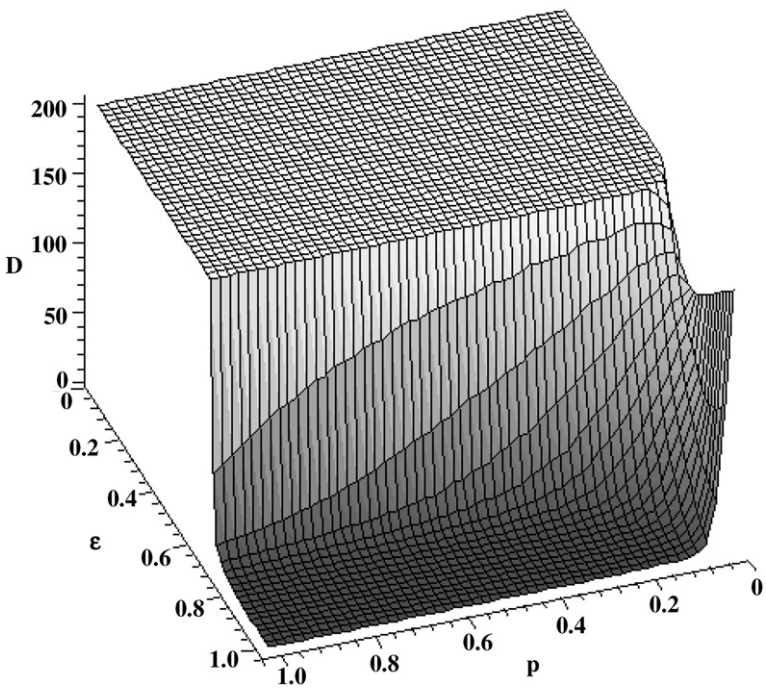

Fig. 2. Lyapunov dimension for a lattice with $N = 201$ maps, $\beta = 3$, and regular and random couplings, as a function of the probability of shortcuts p and the coupling strength ϵ.

positive Lyapunov exponents only. Hence the summation of the Lyapunov dimension may contain negative exponents, as long as the final result is still non-negative. We can thus say that the entropy is more sensitive to variations of sign in the Lyapunov spectrum, as compared with the Lyapunov dimension, which is apparent from Figs. 1 and 2. As the number of positive Lyapunov exponents decrease, both entropy h and Lyapunov dimension D also diminish their values, but the decrease is smoother for h with respect to D. The small values taken on by D indicate that the global attractor of the coupled map lattice becomes of lower dimensionality, i.e., the natural measure is confined to a more restricted phase space region.

As the coupling becomes stronger, both by increasing ϵ (with p fixed) or p (with ϵ fixed), the entropy goes to a small value

$$h = \frac{\lambda_{\mathrm{U}}}{N} = \frac{\ln \beta}{N} \sim \frac{1}{N} \to 0, \quad \text{if } N \to \infty, \tag{7}$$

corresponding to a situation where the Lyapunov dimension D goes to the unity.

This situation has a geometric interpretation. The completely synchronized state (2) defines a one-dimensional synchronization manifold $\mathscr{S}$ embedded in the N-dimensional phase space of the coupled map lattice. The dynamics along $\mathscr{S}$ is the same as of an uncoupled chaotic map with a positive Lyapunov exponent λ_{U}. If we observe such a completely synchronized state, there are $N-1$ remaining negative Lyapunov exponents, associated with transversal directions to the synchronization manifold. Another numerical diagnostics of identical synchronization can also be used, as the Kuramoto's order parameter [23]. When applied to the coupled map lattice with small-world coupling (1), the results are in accordance with those indicated by the KS-entropy [24].

3. Map lattice with power-law coupling

Some properties of lattices with small-world coupling can be understood by resorting to a map lattice with a coupling whose strength decreases with the distance as a power-law in the following form [25]:

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \frac{\epsilon}{\eta(\alpha)} \sum_{j=1}^{N'} \frac{1}{j^{\alpha}} \left[f(x_n^{(i+j)}) + f(x_n^{(i-j)}) \right], \tag{8}$$

where $\epsilon > 0$ and $\alpha > 0$ are the coupling strength and range, respectively, and

$$\eta(\alpha) = 2 \sum_{j=1}^{N'} \frac{1}{j^{\alpha}} \tag{9}$$

is a normalization factor, with $N' = (N-1)/2$ for N odd. The local dynamics will be given, as before, by the same piecewise linear map.

The coupling term in Eq. (8) is a weighted average of discretized second spatial derivatives, the normalization factor being the sum of the corresponding weights. If $\alpha \to \infty$, only those terms with $j = 1$ will contribute to the summations present in the coupling term, which results in $\eta \to 2$, so that we obtain the usual future Laplacian coupling

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \frac{\epsilon}{2} [f(x_n^{(i+1)}) + f(x_n^{(i-1)})], \tag{10}$$

which connects nearest-neighbors only [26,27]. In the case where $\alpha = 0$, we have that $\eta = N-1$, and the coupling becomes of a global type

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \frac{\epsilon}{N-1} \sum_{j=1, j\neq i}^{N} f(x_n^{(j)}), \tag{11}$$

where each site interacts with the mean value of all lattice sites, irrespective of their relative positions [28]. The power-law coupling in Eq. (8) may be regarded as an interpolating form between the nearest-neighbor (local) and mean-field (global) prescriptions.

For these two limiting cases of the coupling prescription (8) the Lyapunov spectrum is analytically known thanks to the constant slope of $f(x)$. For $\alpha \to \infty$ we have the local or diffusive coupling (10). Using periodic boundary conditions, $x_n^{(0)} = x_n^{(N)}$, the Jacobian matrices are symmetric and circulant, and the corresponding Lyapunov spectrum is [29–31]

$$\lambda_j(\beta, \epsilon) = \ln \beta + \ln \left| 1 - \epsilon \left[1 - \cos\left(\frac{2\pi j}{N} \right) \right] \right| \quad (j = 1, 2, \ldots, N). \tag{12}$$

Likewise, for $\alpha = 0$ the global *mean-field* coupling in Eq. (11) leads to the following $(N-1)$-fold degenerate Lyapunov spectrum [28]:

$$\begin{aligned} &\lambda_1(\beta) = \ln\beta, \\ &\lambda_j(\beta,\epsilon) = \ln\left|\beta\left[1-\epsilon\left(1+\frac{1}{N-1}\right)\right]\right| \quad (j = 2,3,\ldots,N), \end{aligned} \tag{13}$$

provided $\epsilon < (N-1)/N$.

We analyze the dependence of the KS-entropy density with the power-law coupling strength ϵ and effective range α. Fig. 3 shows a plot of h versus α and ϵ for a lattice of piecewise linear maps with $\beta = 3$. For a global coupling ($\alpha = 0$) the mean value of the KS-entropy density is close to zero for strong coupling (large ϵ) and at a given critical value $\epsilon^* \approx 0.6$ it grows monotonically to a maximum value, achieved for vanishing coupling, and turns to be just the Lyapunov exponent for uncoupled maps (λ_U).

It is possible to understand this transition based on the Lyapunov spectrum for the global coupling case given by (13) [11]. Except for $\lambda_1 = \ln\beta$, which is always positive for $\beta > 1$, the other $N-1$ exponents are positive provided $\epsilon < \epsilon_c = 1-(1/\beta)$. Hence, for $\epsilon \geqslant \epsilon_c$ the KS-entropy density is usually very small and vanishes for $N \to \infty$. For $0 < \epsilon < \epsilon_c$ all the exponents are positive, and the KS-entropy density depends on ϵ as

$$h(\alpha = 0) = \frac{1}{N}\left\{\ln\beta + (N-1)\ln\left[\beta\left(1-\epsilon-\frac{\epsilon}{N-1}\right)\right]\right\}. \tag{14}$$

For $\beta = 3$ and large N, h goes to zero for $\epsilon \geqslant \epsilon_c = 2/3$, in agreement with the result depicted in Fig. 3.

As the effective range increases, we still have a such transition for some critical coupling strength $\epsilon^*(\alpha)$, after which the entropy increases and achieves its maximum value when the maps are uncoupled. For stronger coupling h takes on positive values that, on its turn, also increase with the range parameter. When α is large the coupling between maps becomes effectively noticeable with nearest-neighbors, and even a strong coupling is not able to change the chaotic dynamics of the orbits for each map, even though the number of positive Lyapunov exponents diminishes as the coupling strength grows.

We can use the analytical result for the Lyapunov spectrum of the local case, Eq. (12), to explain this result. The maximum and minimum Lyapunov exponents are λ_U and $\lambda_{\min}(\epsilon) \equiv \lambda_{N/2}(\epsilon) = \ln[\beta(1-2\epsilon)]$, respectively. If

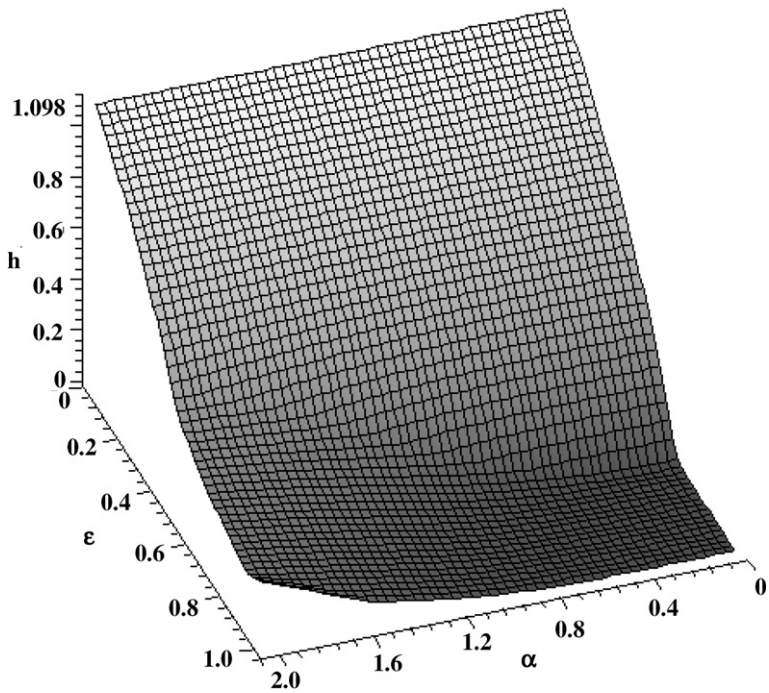

Fig. 3. KS-entropy density in terms of the coupling range and strength, for a lattice of $N = 201$ piecewise linear maps with $\beta = 3.0$ and a power-law coupling scheme.

$\epsilon < \tilde{\epsilon}_c = (1 - (1/\beta))/2$ we have $\lambda_{\min} > 0$ and all exponents are positive, giving a large value for h. Even for $\epsilon \geqslant \tilde{\epsilon}_c$ (=1/3 for $\beta = 3$) there are still typically many positive exponents, what ensures that h be positive, although it nonetheless decreases as ϵ grows. In the case of local coupling, and some particular values of ϵ, it is even possible to derive analytical expressions for the entropy density [31]. Fig. 4 presents our results for the Lyapunov dimension D, whose dependence on the coupling parameters (strength and range) agrees qualitatively with the previous analysis based on the KS-entropy density.

Coupled map lattices of the type treated in this paper, in which there are regular and random connections, exhibit, in the large-p limit, properties similar to those of globally coupled lattices, i.e., when we set $\alpha = 0$ in Eq. (8). Let us consider the coupling given by Eq. (1), in the limit when the connectivity matrix I_{ij} has all its entries equal to the unity, i.e., $M = N$, what gives $p = 1$. This means that all lattice sites have N non-local shortcuts and thus $k = N + 4$ connections at all. Notice that in this case the nearest and second-nearest neighbors are counted twice. This has a negligible effect on the final results, and we must borne in mind that the small-world coupling treated in this paper is supposed to be typically applied for *small p*.

On the other hand, the power-law coupling (8) implies that every site is coupled with all other sites of the lattice, what means that there are always $k = N$ connections per site. Moreover, for $\alpha = 0$, we get a global coupling in which all sites interact in the same way regardless of the distance along the lattice. In the same way, our small-world coupling prescription supposes that both local and non-local interactions have the same strength. On this basis, we conclude that the small-world coupling in the limit $p \to 1$, although not exactly equivalent, has practically the same properties as a global coupling obtained in the $\alpha \to 0$ limit of a power-law coupling. This explains the similarity between the behavior of the entropy and Lyapunov dimension for both coupling schemes.

A further manifestation of this analogy between small-world and mean-field lattices is that, while the spectrum of Lyapunov exponents for a globally coupled lattice as (11) exhibits a transition to low-dimensional chaos at a critical value of $\epsilon_c = 2/3$ [11], we found a similar transition for our lattice, when p is comparatively large, at a value around ϵ_c. On the other hand, when p vanishes, only the nearest and the next-to-the-nearest neighbors will contribute to coupling in (1), and we obtain essentially the local diffusive coupling form (10). Hence, the shortcut probability plays a role similar to the range parameter, interpolating the coupling prescription between the global and local limits. We stress, however, that this analogy has obvious limits of applicability, since the nature of the interactions are quite distinct.

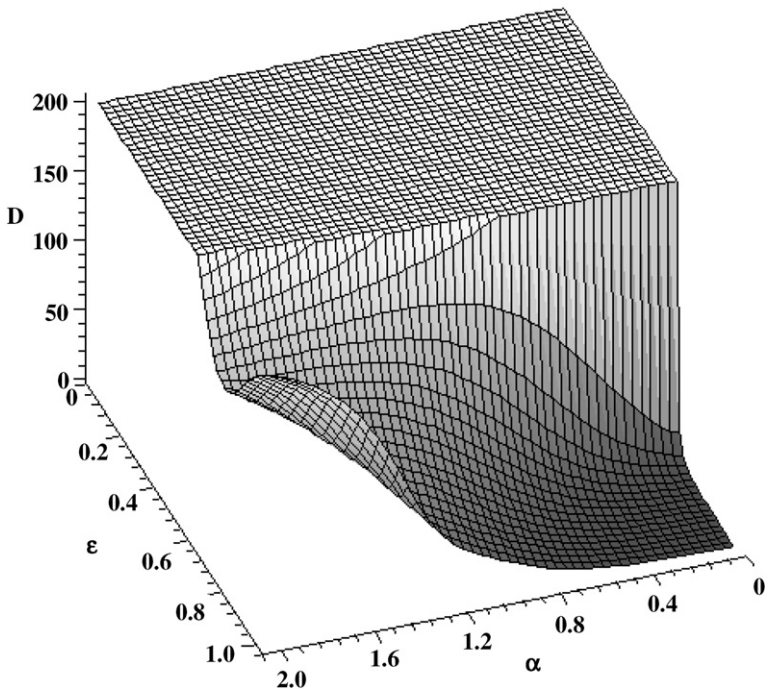

Fig. 4. Lyapunov dimension for a lattice with $N = 201$ maps, $\beta = 3$, a power-law coupling, as a function of the range parameter α and the coupling strength ϵ.

4. Conclusions

In this paper we explored some aspects of the spatio-temporal dynamics displayed by a coupled map lattice with regular and random interactions, inspired in models proposed for describing small-world networks. From the dynamical point of view, our numerical results indicate that the chaoticity of the lattice as a whole, as measured by its Lyapunov spectrum, essentially decreases as the number of random shortcuts is increased, indicating that the addition of non-local connections to an otherwise locally coupled chaotic lattice has the ability of reduce chaos in the system. The reduction of chaoticity of the lattice suggests that the dynamics exhibits completely synchronized states, for which the system attractor is a one-dimensional manifold possessing the dynamics of an isolated system. These findings are in accordance with analyzes of numerical diagnostics of synchronization. Moreover, the qualitative features of the lattices with random networks and a large shortcut probability are similar to globally coupled lattices, for which a given site interacts with the mean-field of all other sites, irrespective of their lattice positions.

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