



Noise-induced basin hopping in a gearbox model

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Abstract

We investigate some dynamical effects of adding a certain amount of noise in a theoretical model for rattling in single-stage gearbox systems with a backlash, consisting of two wheels with a sinusoidal driving. The parameter intervals we are dealing with show an extremely involved attraction basin structure in phase space. One of the observable effects of noise is basin hopping, or the switching between basins of different attractors. We characterize this effect and its relation to the presence of chaotic transients.

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1. Introduction

There is a growing interest in the dynamical study of vibroimpact problems [1]. First, vibroimpact systems are widely found in engineering applications, like vibration hammers, driving machinery, milling, impact print hammers, and shock absorbers [2,3]. There are also undesirable effects coming from vibroimpact systems like gearboxes, bearings, and fuel elements in nuclear reactors: large amplitude response leading to material fatigue and rattling [4]. On the other hand, vibroimpact systems are also of interest by themselves, for they are non-smooth dynamical systems which usual mathematical methods, like bifurcation theory, are applicable only to a limited extent [5–7].

Gear units have typically backlashes, or variable clearances between adjacent moving parts, and these backlashes are needed to allow thermal expansion and lubrication of the moving wheels. The presence of backlashes makes gear teeth lose contact for a short interval and collide again producing a hammering effect [8,9]. In spur gears of engines driving the camshafts and the injection pump shafts, for example, rattling is the source of uncomfortable noise. Hence, many theoretical and experimental efforts have been devoted to the understanding of such vibroimpact problems [10–12].

We focus on the gearbox rattling model proposed by Pfeiffer and collaborators, and consisting of two spur gears with different diameters and a gap between the teeth (single-stage rattling) [8,9,13]. The motion of one gear is supposed

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to be a sinusoidal motion with well-defined amplitude and frequency, whereas the motion of the other gear results from the system dynamics [9,13].

Previous numerical studies on this model have shown a rich dynamical behavior for the problem, as coexisting periodic and chaotic attractors, with fractal basins of attraction [14]. As these chaotic attractors collide with their own basin boundaries, or with unstable periodic orbits, there are crises which suddenly alter the nature or the size of the chaotic attractor [15]. There has been found that control of chaotic dynamics is also possible by means of stabilizing unstable periodic orbits embedded in the chaotic attractor [16,17]. Non-ideal energy sources (with a limited power-supply) have also been studied from the gear rattling point of view [18].

In this paper we aim to investigate some dynamical effects of noise in this gear-rattling model, since extrinsic noise is unavoidable in laboratory and industrial contexts. One of the outstanding features we describe is a phenomenon called *basin hopping*, and which consists of the intermittent switching between two or more basins of attraction, when the system is subjected to noise. Basin hopping is conspicuous, in particular, when the basin boundaries are fractal. In practice basin hopping can be highly undesirable and even dangerous, since it may produce large amplitude jumps in the motion of the gear wheels and a consequent fatigue of the material, if not the complete collapse of the gearbox while in operation.

This paper is organized as follows: Section 2 presents the basic equations governing the gear-rattling model we are investigating. Section 3 presents some aspects of the basin boundary structure without noise, showing that the basins of attraction present an extremely involved structure. Section 4 deals with the basin hopping effect induced by external noise applied to the system, and also with the relation of this effect with the presence of chaotic transients. Our conclusions are left to the last section.

2. Pfeiffer's gear-rattling model

A single-stage gearbox system will be represented by two gears with radii R and R_e , and with a backlash v between the teeth, measured in the mesh plane of the gears (see Fig. 1) [8,13]. The motion of the driving wheel is supposed to be a harmonic oscillation, whereas the second gear has its dynamics resulting from the repeated impacts between the two teeth. Between the impacts, the motion of the second wheel is governed by a linear differential equation and can be analytically determined. The impacts are treated by modifying the initial conditions of the motion, according to the Newtonian impact law [19].

In the absolute coordinate system, we denote by φ be the angular displacement of the second gear, such that the rotation dynamics is governed by equation of motion [8,9,13,19] (primes denote differentiation with respect to time):

$$m\varphi'' + d\varphi' = -T, \quad (1)$$

where m is the moment of inertia, d is the oil drag coefficient, and T is the oil splash torque. The driving gear undergoes a harmonic rotational motion with angular amplitude A and frequency ω described by $e(t) = -A \sin(\omega t)$. The relative displacement between the gears due to the backlash is thus

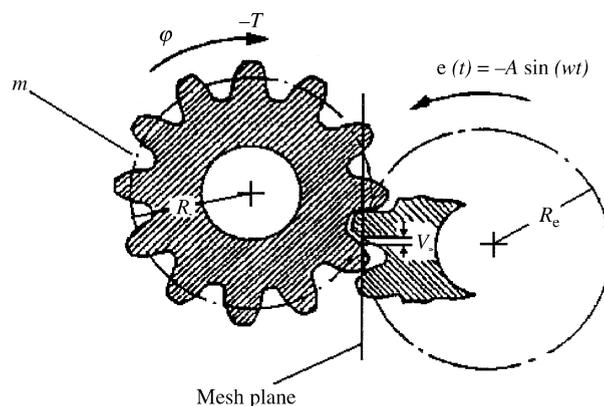


Fig. 1. Schematic view of a single-stage gearbox system.

$$s = \frac{AR_e}{v} \sin \omega t - \frac{R}{v} \phi, \tag{2}$$

in such a way that $-1 < s < 0$.

The equation of motion, in the relative coordinate system, is obtained from (1) and (2), resulting in

$$\ddot{s} + \beta \dot{s} = \ddot{e} + \beta \dot{e} + \gamma, \tag{3}$$

where dots stand for derivatives with respect to the scaled time $\tau = \omega t$, and we have introduced the following non-dimensional parameters

$$\alpha \equiv \frac{AR_e}{v}, \tag{4}$$

$$\beta \equiv \frac{d}{m\omega}, \tag{5}$$

$$\gamma \equiv \frac{TR}{m\omega^2}, \tag{6}$$

representing the damping coefficient, excitation amplitude, and moment of inertia, respectively.

Since Eq. (3) is linear, we can integrate it analytically between two impacts of the gear teeth, after specifying the initial conditions: $s(\tau_0) = s_0$ and $\dot{s}(\tau_0) = \dot{s}_0$. The displacement s and velocity \dot{s} between impacts will be given by:

$$s(\tau) = s_0 + \alpha(\sin \tau - \sin \tau_0) + \frac{\gamma}{\beta}(\tau - \tau_0) + \frac{1}{\beta} \{1 - \exp[-\beta(\tau - \tau_0)]\} \left(\dot{s}_0 - \alpha \cos \tau_0 - \frac{\gamma}{\beta} \right), \tag{7}$$

$$\dot{s}(\tau) = \alpha \cos \tau + \left(\dot{s}_0 - \alpha \cos \tau_0 - \frac{\gamma}{\beta} \right) \exp[-\beta(\tau - \tau_0)] + \frac{\gamma}{\beta}. \tag{8}$$

We have an impact whenever $s = -1$ or 0 , since they actually correspond to the backlash boundaries. At these points the motion is no longer smooth and we have to reset the initial conditions, according to the laws of inelastic impact:

$$\tau_0 = \tau, \tag{9}$$

$$s_0 = s, \tag{10}$$

$$\dot{s}_0 = -r\dot{s}, \tag{11}$$

where $0 < r < 1$ is the restitution coefficient [19].

Adopting the instant of each collision as the time unit, we can define discrete variables s_n , \dot{s}_n , and τ_n representing the displacement, velocity, and time (*modulo* 2π) just before the n th impact. Thanks to the analytical solution of the equations of motion between two impacts, we can substitute the initial conditions $\tau_0 = \tau_n$, $s_0 = s_n$, and $\dot{s}_0 = -r\dot{s}_n$ into Eqs. (7) and (8). With this representation we have a mapping relating the dynamical variables for the $(n + 1)$ th impact to their corresponding values just before the n th impact [14]:

$$s_{n+1} = s_n + \alpha(\sin \tau_{n+1} - \sin \tau_n) + \frac{\gamma}{\beta}(\tau_{n+1} - \tau_n) - \frac{1}{\beta}(1 - \exp[-\beta(\tau_{n+1} - \tau_n)]) \left(r\dot{s}_n + \alpha \cos \tau_n + \frac{\gamma}{\beta} \right), \tag{12}$$

$$\dot{s}_{n+1} = \alpha \cos \tau_{n+1} + \frac{\gamma}{\beta} - \exp[-\beta(\tau_{n+1} - \tau_n)] \left(r\dot{s}_n + \alpha \cos \tau_n + \frac{\gamma}{\beta} \right). \tag{13}$$

Since s_n can assume only the values 0 and -1 , we can follow system trajectories in a two-dimensional phase plane (\dot{s}_n, τ_n) . This map is particularly useful when computing Lyapunov exponents, since due to the non-smoothness of our dynamical systems we cannot use many algorithms of Lyapunov exponent calculation [20].

3. Noiseless dynamics

As a representative example of the kind of dynamics generated by the gearbox system, we depict in Fig. 2 a bifurcation diagram for the time τ_n of the n th impact versus the control parameter α , which is related to the oscillation amplitude of the driving gear. We adopt the following numerical values for this and the forthcoming simulations: $\beta = 0.1$, $\gamma = 0.1$, and $r = 0.95$.

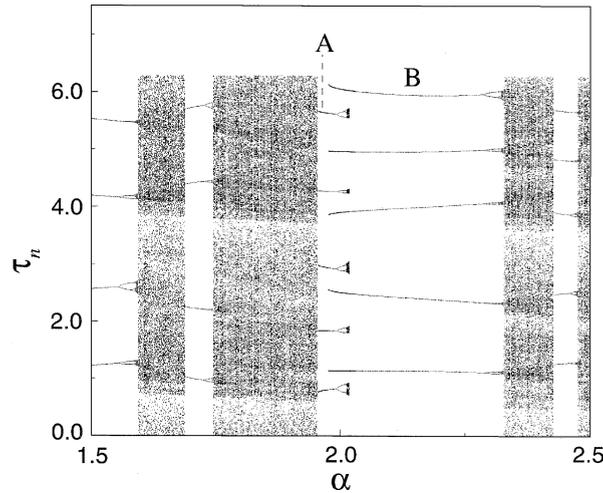


Fig. 2. Bifurcation diagram for the time of impact in terms of the oscillation amplitude.

There are wide intervals for which the system presents chaotic dynamics, interspersed with periodic windows. The attractor labeled as *B* appears in Fig. 2 as the successive times for five impacts, but it nevertheless represent a period-1 orbit. The interval [1.5, 2.5] for the control parameter was chosen to evidence the presence of crises, which are sudden changes in chaotic attractors caused by collisions between the attractors with other structures like basins of attraction, other chaotic attractors or unstable periodic orbits [21].

The periodic attractor indicated by letter *A* in Fig. 2 appears after a boundary crisis, which occurs since the pre-critical chaotic attractor collides with its own basin boundary. The post-critical behavior is characterized as a chaotic transient, since the trajectory experiences an irregular motion very similar to the pre-critical one, but eventually escapes out of this attractor remnant and settles down in a stable period-1 orbit. This orbit, in its own hand, experience a period-doubling cascade which rapidly leads to chaotic motion (barely shown in Fig. 2 due to finite graphical resolution), which suddenly disappears due to a boundary crisis, in which this short-lived chaotic attractor collides with a period-1 unstable orbit.

Other feature displayed by Fig. 2 is the coexistence of the two attractors, *A* and *B*, in the periodic window existing near $\alpha = 2.0$. The dynamics of this gear-rattling system can be also appreciated in continuous time, as shown by the two panels of Fig. 3 in which we show the orbits corresponding to the attractors *A* [Fig. 3(a)] and *B* [Fig. 3(b)]. The Lyapunov exponents were computed for the two attractors (through the use of the mapping equations, which overcome the difficulty with non-smoothness of the trajectory [20]), resulting in negative values very close from each other, namely $\lambda_1 = 0.1140$ and $\lambda_2 = -0.1141$ for attractor *A* [Fig. 3(c)]; and $\lambda_1 = -0.1140$ and $\lambda_2 = -0.1142$ for *B* [Fig. 3(d)].

We can fix a value $s_0 = -0.5$ and work with a two-dimensional section of the phase space. It turns out that the basin structure of the attractors *A* and *B* shown in Fig. 2 is very complex, as revealed by Fig. 4(a), where the basin of attractor *A* (*B*) is depicted in black (white), for a grid of 600×600 pixels. There are regions belonging to both basins interspersed through almost the entire portion of the phase space section in finer scales, as can be seen in a magnification of a small box [Fig. 4(b)].

The intertwined nature of the basins of *A* and *B* suggests the fractality of the corresponding basin boundary. In order to check this claim and make quantitative statements we have used the uncertainty exponent technique, consisting on determining the fraction of uncertain initial conditions in the phase space section we considered in Fig. 4. To each initial condition (\dot{s}_0, τ_0) leading to some attractor we applied small ϵ -perturbations $(\dot{s}_0 \pm \epsilon, \tau_0)$ and checked out whether or not the perturbed initial condition asymptotes to the same attractor as the unperturbed one. If it does not so, the initial condition is dubbed as uncertain, and we determine the fraction $f(\epsilon)$ of them with respect to the total number of grid points. It is well-known that this fraction scales with ϵ as a power-law $f(\epsilon) \sim \epsilon^\varpi$, where ϖ is the uncertainty exponent [22]. Smooth basin boundaries are characterized by $\varpi = 1$, whereas fractal ones to smaller values for the uncertainty exponent.

Fig. 5 displays how the uncertain fraction scales with ϵ for the basin structure of Fig. 4(a), for 3000 randomly chosen initial conditions. The slope of the fitted line (in a log–log plot) gives us an uncertainty exponent of $\varpi = 0.0139 \pm 0.0005$. Besides being fractal, the fact that ϖ is near zero indicates that the basins are so intertwined that it becomes extremely

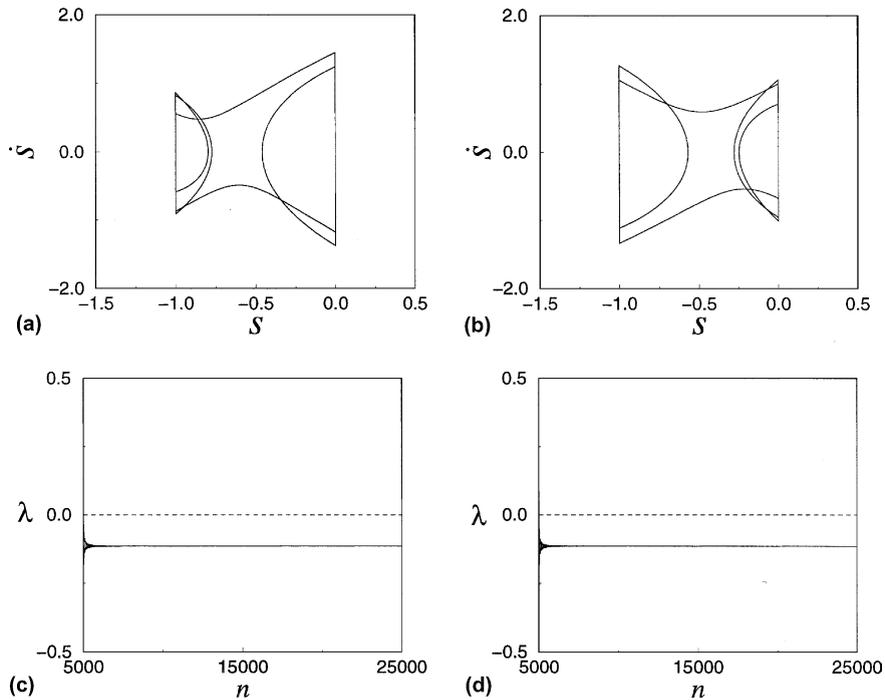


Fig. 3. Phase portrait of velocity versus displacement for $\alpha = 1.985$ showing coexisting attractors A (a) and B (b). The time series of the corresponding Lyapunov exponents are shown in (c) and (d), respectively.

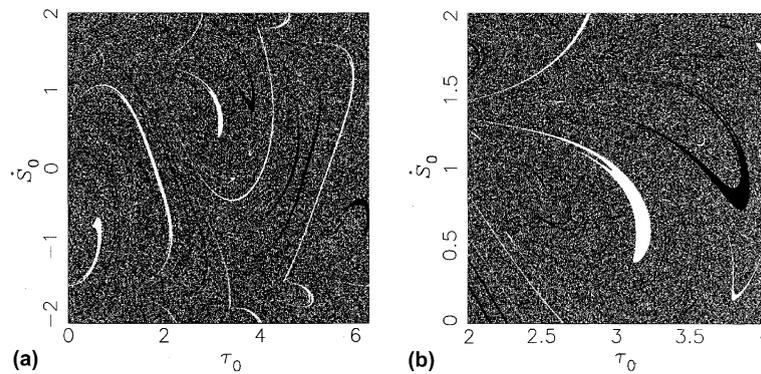


Fig. 4. (a) Basins of attractors A (black) and B (white) in a phase space section of initial conditions for the velocity versus time of impact and (b) magnification of a portion of the previous figure.

difficult to predict to what attractor the system will asymptote to, since ϵ can be regarded as the error in specifying the initial condition. If we increase, for example, the accuracy by a factor of one hundred (i.e., $\epsilon \rightarrow \epsilon = \epsilon/100$), the corresponding uncertain fraction will decrease the modest amount of 6.2%. This has been called by Grebogi et al. [22] as *final-state sensitivity*, and represents an obstacle to prediction of seriousness only comparable with chaos itself. In mechanical systems, this kind of phenomena has been called *practical riddling* [23].

4. Noisy dynamics

The effects of extrinsic noise on chaotic dynamics have been studied since the early days of non-linear research [24]. For systems displaying period-doubling bifurcations the presence of noise leads to both a gap in the bifurcation

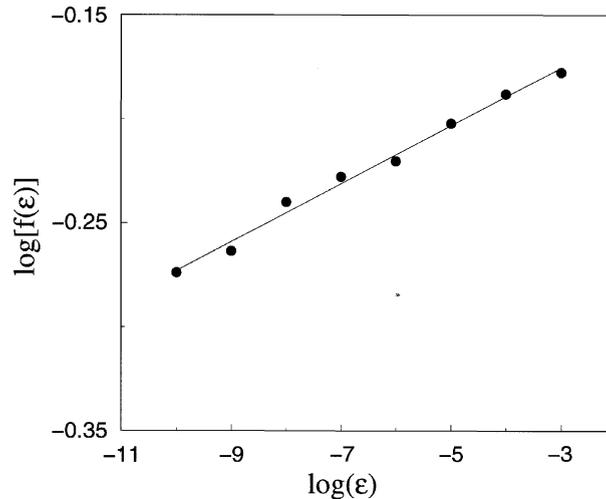


Fig. 5. Uncertain fraction of initial conditions versus the uncertainty radius for the basin structure of Fig. 4(a). The straight line is a least squares fit.

sequence, additional bifurcation, as well as a renormalization of the chaotic threshold [25]. In driven non-linear oscillators increased noise levels may induce a transition to chaotic behavior or, if chaos already exists, noise can enhance chaotic behavior, while destroying periodic orbits [26].

Noise in dynamical systems can be regarded in two basic ways: (i) an external stochastic force that perturbs the phase space trajectory; and (ii) a random perturbation of the system parameters [25]. These are commonly called additive and parametric noise, respectively. Although, in a laboratory context, both kinds of noise are present, in this paper we restrict ourselves with parametric noise applied to the control parameter α , since we are focusing on the effects of noise as this control parameter is varied. Hence we will consider

$$\alpha \rightarrow \alpha(1 + \sigma p_n), \quad (14)$$

where p_n represents a uniform random variable on the interval $(-1, +1)$ with unit variance and zero mean, and σ will be referred to as the noise level of the parametric fluctuations. In terms of the mathematical model we are dealing with in this paper, the σp_n term is applied just after the moment of the impact between gears. This is physically reasonable since between impacts the gears are effectively uncoupled.

Let us consider the effects of such noise in the dynamical scenario described in the previous section, where we have seen a multi-stable structure, with coexisting periodic (and, perhaps, also chaotic) attractors whose basins are complexly interwoven. The basins are so intertwined, as revealed by the extremely small value of the uncertainty exponent ϖ , that the basin boundary dimension is very close to the phase space dimension. We can think of ϵ as being the noise level applied to a phase space trajectory at a given time. As a result, a trajectory is constantly being pushed away from or towards a given attractor in an intermittent fashion, hindering long-time predictions. If there are two or more attractors, the trajectory may pass certain amount of time in each attractor, jumping to other ones as time evolves. This phenomenon has been called *basin hopping*.

This fact is clearly seen in the time series of one of the dynamical variables (with similar behavior displayed by the other ones) shown in Fig. 6(a)–(d), with increasing noise levels. Fig. 6(a), with 0.5% of noise added, shows no noticeable effect for attractor A , at least over the first 50,000 map iterations (remember that this is a period-1 orbit, corresponding to five successive impacts). As the noise level is increased to 0.8% [Fig. 6(b)] the basin hopping is clearly visible, since the trajectory visits alternately the attractors A and B . The average time the trajectory stays in each attractor diminishes with increasing noise level [Fig. 6(c)], and can be so short that it could lead to the erroneous conclusion that the attractor would be chaotic [Fig. 6(d)].

The basin hopping phenomenon is being currently a topic of active research. The fact that the length of the intervals the trajectory stays in each attractor varies irregularly and depends on the noise amplitude reveals a close relation with the so-called chaotic itinerancy [27], which has also been observed experimentally [28]. Such noise-induced phase transitions have also been observed in coupled oscillator systems with delay [30].

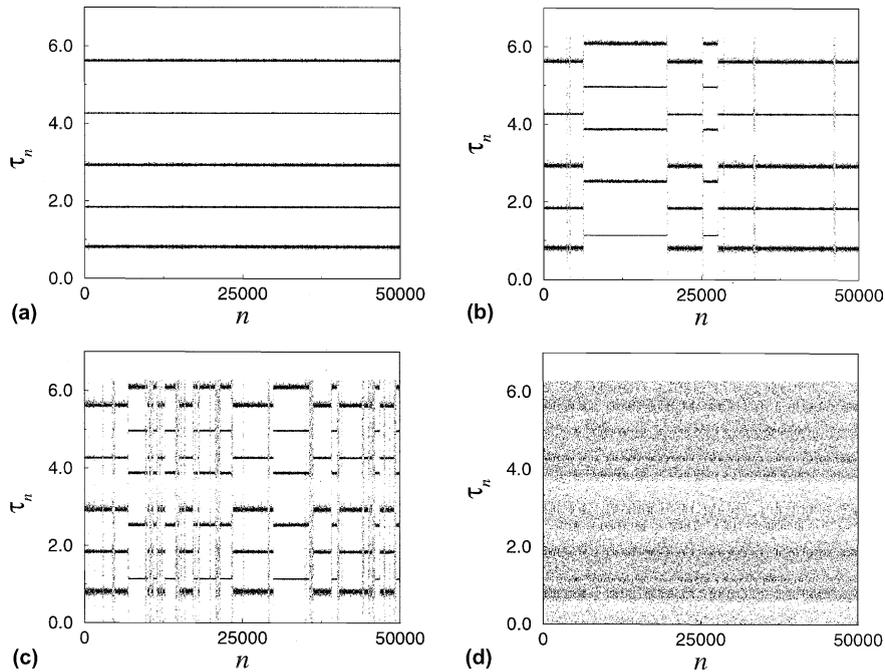


Fig. 6. Time series for the τ_n variable for the attractor A , for $\alpha = 1.985$, and increasing noise levels: (a) $\sigma = 0.005$; (b) 0.008 ; (c) 0.010 ; and (d) 0.035 .

The theoretical explanation of basin hopping is based in the properties of the chaotic saddles which lie in between the basins. The noise term kicks the trajectory out of an open neighborhood of some attractor into the basin boundary. The trajectory spends there a certain amount of time until it reaches again the neighborhood of the same or another attractor. The chaotic saddle underlying the basin boundary structure is a non-attracting chaotic invariant set, so that the trajectory, while wandering in the vicinity of the saddle, experiences a transient chaotic motion. The dependence of the time a trajectory spends in each basin is due to a competition between the attractiveness toward regular motion in the neighborhood of an attractor and the jumping among the different attractors induced by the noise [29]. The presence of chaotic transients is thus an observable manifestation of the chaotic saddle existing in between the basins of attraction. Another consequence of the chaotic saddles is the practical riddling phenomenon observed in the basin structure [31].

In Fig. 7(a) we depict the periodic attractor A (in a section of the phase space) with a very tiny amount of noise (0.5%), such that there is a barely noticeable effect in the attractor itself, other than some blurring of the periodic orbit, or a small wiggling of the corresponding Lyapunov exponents [Fig. 7(c)]. However, when a higher noise level is applied [Fig. 7(b)], the system is expected to visit both attractors A and B due to basin hopping, and also to display a chaotic transient while the trajectory is under the influence of the chaotic saddle. In fact, the chaotic transient (which is superimposed in the figure to the two coexisting periodic attractors) resembles the chaotic attractor that would exist for the noiseless system at a *different* parameter value. This is actually the case, as revealed in Fig. 8(a), where we depict (in a similar phase space section) the chaotic attractor the system displays for $\alpha = 1.95$.

Since the transient dynamics persists indefinitely, as long as noise is continually being applied to the systems, the chaotic transient acquires a stationary character, as confirmed by the positiveness of one of the Lyapunov exponents [Fig. 7(d)]. The positive Lyapunov exponent of the noisy chaotic transient (0.3742) is only slightly less than the maximal Lyapunov exponent of the noiseless chaotic attractor (0.4462), the difference being a result of the different parameter values corresponding to these cases.

The resemblance between the chaotic attractor and the chaotic transient can be regarded as the result of the boundary crisis suffered by the system at $\alpha \approx 1.96$. The pre-critical chaotic attractor of Fig. 8 (at $\alpha = 1.95$) collides with its own basin boundary and disappears, becoming a chaotic non-attracting saddle. For post-critical α -values not far from the crisis value the chaotic saddle resembles the chaotic attractor and, as depicted in Fig. 7(b), yields a chaotic transient very similar to the pre-critical chaotic attractor. The chaotic transient has a finite duration before evolving through one of the coexisting post-critical periodic attractors, but it is actually sustained by the noise effect.

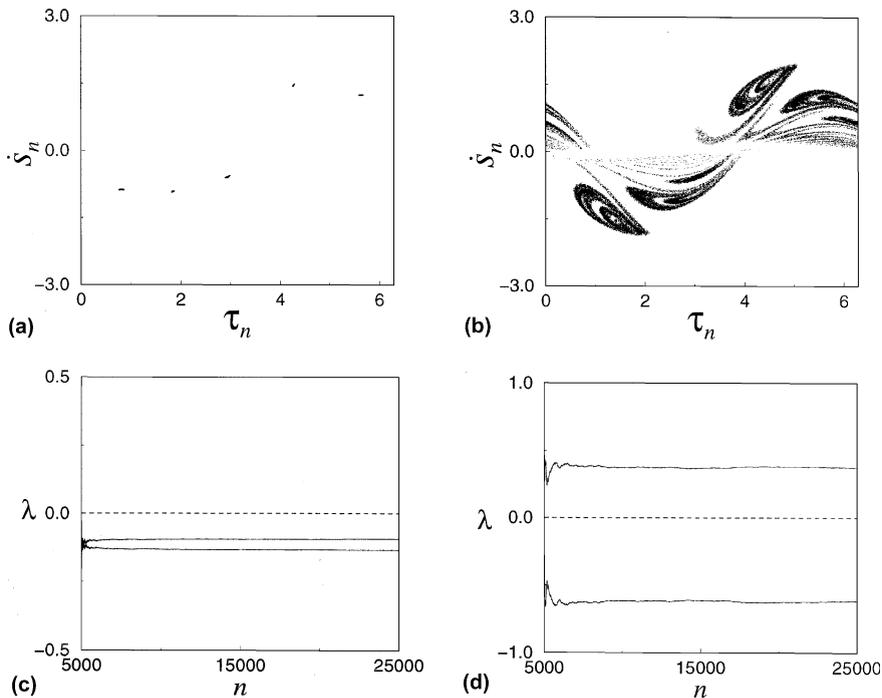


Fig. 7. Phase portrait of velocity versus impact time for the attractor A with $\alpha = 1.985$ and increasing noise levels: (a) $\sigma = 0.005$; (b) 0.035. The corresponding Lyapunov exponents are represented, as a function of the number of impacts, in (c) and (d), respectively.

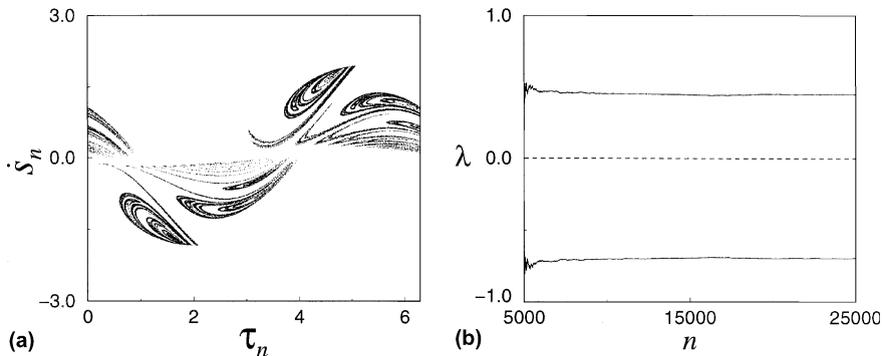


Fig. 8. (a) Phase portrait (velocity versus impact time) for $\alpha = 1.95$ (before the crisis) and no noise and (b) time series of the corresponding Lyapunov exponents.

5. Conclusions

In this paper we investigated some effects of parametric noise in the dynamics of a gear-rattling model. Since the noiseless dynamics is found to exhibit a complex attraction basin structure, the addition of small amount of noise provokes basin hopping, or the alternate switching among different coexisting attractors (periodic or chaotic), or basin-hopping. The alternations occur in different time intervals whose average length decreases as the noise level builds up. Moreover, while making transitions between different attractors, the trajectory experiences a chaotic transient.

This chaotic transient is related to the chaotic saddle existing in between the coexisting basins and, in presence of noise, can be sustained to yield an apparently stationary regime very similar to a noiseless chaotic attractor. The relation between these regimes, for the parameter interval explored in this paper, is a boundary crisis phenomenon which transforms a pre-critical chaotic attractor into a post-critical chaotic saddle.

In terms of the gear-rattling mechanical system which our model purports to describe, it is important to know how much noise can be applied to the system such that basin hopping is not discernible. Basin hopping can be dangerous, if one of the coexisting (periodic or chaotic) attractors lead to oscillation amplitudes large enough to cause harm to the gearbox or other coupled mechanisms. We have found that a parametric noise level of less than 0.5% does not lead to basin hopping and can be a safe limit to the gearbox operation.

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