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Unstable dimension variability and codimension-one bifurcations of two-dimensional maps

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Abstract

Unstable dimension variability is a mechanism whereby an invariant set of a dynamical system, like a chaotic attractor or a strange saddle, loses hyperbolicity in a severe way, with serious consequences on the shadowability properties of numerically generated trajectories. In dynamical systems possessing a variable parameter, this phenomenon can be triggered by the bifurcation of an unstable periodic orbit. This Letter aims at discussing the possible types of codimension-one bifurcations leading to unstable dimension variability in a two-dimensional map, presenting illustrative examples and displaying numerical evidences of this fact by computing finite-time Lyapunov exponents.

Hyperbolic invariant sets, like chaotic attractors or chaotic saddles, play a major role in the theory of dynamical systems, thanks to many convenient mathematical properties, such as:

(i) The stable and unstable manifolds can be defined for each point belonging to the set [1,2];
(ii) The set and the corresponding dynamics are structurally stable, i.e., small perturbations do not topologically alter its dynamics [3];
(iii) Noisy trajectories of hyperbolic systems are closely followed by fiducial (noiseless) trajectories of the system for an infinite time [4,5].

Unfortunately, most dynamical systems of physical interest fail to be hyperbolic, thus limiting the applicability of hyperbolicity to only a few models, like Axiom-A systems and topological horseshoes [2].

When the decomposition of the tangent space into a stable and unstable subspace does not vary continuously along the invariant set, the dimension of the unstable manifold may be generally different for distinct points belonging to the set. This has been called unstable dimension variability (UDV), and its presence is a severe violation of the necessary conditions for a set to be hyperbolic [6]. A particularly troublesome consequence of UDV is the lack of adequate shadowability

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properties of noisy trajectories, such as those obtained by using computers, where the role of noise is played by unavoidable one-step roundoff and truncation errors.

If UDV is too severe, it may happen that a noisy trajectory is not closely followed by *any* fiducial trajectory of the original system for a reasonable time. Hence, the computer-generated trajectories in this case may be just numerical artifacts, and no relevant statistics can be extracted from such orbits [7]. In this case, even though the system is formally a deterministic one, the character of the orbits is, at best, of a stochastic system. Hence, they could be more properly referred to as *pseudo-deterministic systems*. In fact, when there is no shadowability at all, the mathematical model itself may be of limited use, and one should resort to experimental data (using embedding techniques, for example) to obtain relevant information about the system dynamics.

UDV was first described [6] for a diffeomorphism in the space $\mathsf{T}^2 \times \mathsf{S}^2$. The earliest observation of UDV for a dynamical system of physical interest was reported for the kicked double rotor map [8,9]. The presence of UDV seems to be typical in high-dimensional dynamical systems, as in coupled map lattices [10]. Besides their own importance as models of complex systems, they may also appear in numerical methods for solving partial differential equations. Low-dimensional systems, however, also may present UDV, frequently with a non-attractive invariant set, as a chaotic saddle [11]. The relation between UDV and riddled basins, as well as with on–off intermittency, in cases where there is an invariant subspace on which the chaotic set lies, has been discussed in Refs. [12] and [13], respectively.

Although the mechanism for the emergence of UDV in coupled quadratic maps has been studied [14], we still do not have a complete understanding of under what circumstances UDV appears in a dynamical system, as a parameter is varied. It seems natural, though, that this onset must be triggered by some bifurcation of a fixed point or periodic orbit, in the sense that some formerly stable eigendirection becomes unstable, augmenting the dimension of the unstable subspace by one unit. The purpose of this Letter is to enumerate the local codimension-one bifurcations which lead to UDV in two-dimensional maps [1,15]. They were found to mark the onset of UDV for many of the dynamical systems studied so far [8,12,16]. We will restrict our analysis to maps, bearing in mind that they can also be considered as Poincaré sections or time-T stroboscopic samplings of continuous time flows.

It is convenient here to state some basic definitions for further reference. Let $\mathbf{f}: \mathsf{R}^d \to \mathsf{R}^d$ be a d-dimensional map. The set $\Lambda \subset \mathsf{R}^d$ is invariant under $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ if, for any $\mathbf{x}_0 \in \Lambda$, we have $\mathbf{f}^n(\mathbf{x}_0) \in \Lambda$ for all n. The invariant set Λ is said to be hyperbolic if the tangent space $T_\mathbf{x}$ associated with any point $\mathbf{x} \in \Lambda$ can be decomposed into the direct sum $T_\mathbf{x} = E^u_\mathbf{x} \oplus E^s_\mathbf{x}$, where E^u (E^s) is the unstable (stable) subspace, such that the following conditions hold [1]:

(i) The splitting $E^u_\mathbf{x} \oplus E^s_\mathbf{x}$ varies continuously with $\mathbf{x} \in \Lambda$ and is invariant insofar as $\mathbf{Df}(E^u_\mathbf{x}) = E^u_{\mathbf{f}(\mathbf{x})}$ and $\mathbf{Df}(E^s_\mathbf{x}) = E^s_{\mathbf{f}(\mathbf{x})}$, where $\mathbf{Df}$ is the Jacobian derivative. In other words, one finds in Λ continuously varying bases for $E^u_\mathbf{x}$ and $E^s_\mathbf{x}$.

(ii) Forward (backward) iterates of points belonging to the stable (unstable) subspace are attracted to the point $\mathbf{x}$ as n goes to infinity, with an exponential rate ρ which is uniform for all $\mathbf{x} \in \Lambda$. Hence, there exists $K > 0$ and $0 < \rho < 1$ such that $\|\mathbf{Df}^n(\mathbf{x})\mathbf{y}\| < K\rho^n\|\mathbf{y}\|$ if $\mathbf{y} \in E^s_\mathbf{x}$ and $\|\mathbf{Df}^{-n}(\mathbf{x})\mathbf{y}\| < K\rho^n\|\mathbf{y}\|$ if $\mathbf{y} \in E^u_\mathbf{x}$. The unstable (stable) dimension $d^u_\mathbf{x}$ ($d^s_\mathbf{x}$) is the dimension of the invariant unstable (stable) subspace.

The structural stability of hyperbolic maps rules out any qualitative change of periodic orbits due to bifurcations or crises, for example. Since these changes are expected for most dynamical systems of physical interest, it follows that they are not typically hyperbolic [3]. Hence, we concentrate ourselves on how a given dynamical system loses hyperbolicity, and what could be the consequences for this fact. One of the consequences leads to the loss of shadowability for non-hyperbolic noisy orbits, which implies that we cannot take for granted that computer-generated orbits of non-hyperbolic systems are shadowed, or closely followed, by true orbits for an arbitrarily long time.

There are basically two mechanisms for losing hyperbolicity. The first one occurs when there are points on the invariant chaotic set Λ where the stable and unstable manifolds intersect tangentially (homoclinic tangencies). At those tangency points, the invariant subspaces $E^u_{\mathbf{x}}$ and $E^s_{\mathbf{x}}$ are undefined. Once a given homoclinic tangency occurs at a given point, each iterate of this point under the map $\mathbf{f}$ is also a tangency. Even in this case, it is possible to get fiducial chaotic trajectories which shadow computer-generated ones, provided we are far enough from a tangency. The time τ it takes for a noisy trajectory to reach a quasi-tangency, or "glitch", is roughly the time-span of a shadowing trajectory. If $\varpi = \max[\varpi(t)]$ is the noise level corresponding to one-step errors, the shadowing time of a noisy-trajectory is of the order $\varpi^{-\alpha}$, where $\alpha \lesssim 1/2$ is the scaling exponent [17].

We are concerned, however, with a second and more severe way to lose hyperbolicity (UDV), by which the splitting $E^u_{\mathbf{x}} \oplus E^s_{\mathbf{x}}$ does not vary continuously for all points of a chaotic invariant set Λ, because the dimension of the invariant subspace takes on different values for points in Λ. For simplicity, we consider a one-parameter family of two-dimensional maps $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \rho)$, where $\mathbf{x} \in \mathsf{R}^2$ and $\rho \in \mathsf{R}$. Let us assume that, for some value of the parameter, $\rho = \rho_0$, this map has a chaotic invariant set Λ with an unstable fixed point $\mathbf{p} = \mathbf{f}(\mathbf{p}, \rho_0)$.

We say, without loss of generality, that Λ and the dynamics on this set present UDV if there are at least two fixed points in Λ, $\mathbf{p}_1$ and $\mathbf{p}_2$, such that $d^u_1 = \dim E^u(\mathbf{p}_1) \neq d^u_2 = \dim E^u(\mathbf{p}_2)$ for $\rho = \rho_0$. In two dimensions, when $d^u = 1$ (2), $\mathbf{p}$ is a saddle point (repeller), such that $d^u_2 = d^u_1 + 1$. Since every pre-iterate of both fixed points has the same unstable dimension as of $\mathbf{p}_1$ and $\mathbf{p}_2$ themselves, there are two sets of eventually fixed points in Λ of different unstable dimensions, characterizing thus UDV at ρ_0. Let us also assume, without loss of generality, that there exists $\delta\rho_2 > \delta\rho_1 > 0$ such that in the interval $[\rho_0 - \delta\rho_1, \rho_0 + \delta\rho_2]$ we have $d^u_1 = d^u_2$ if $\rho < \rho_0$ and $d^u_2 = d^u_1 + 1$ otherwise. The onset of UDV in this case is a codimension-one bifurcation at $\rho = \rho_0$ of an unstable fixed point embedded in Λ. It may also happen that a period-q orbit $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_q\}$ undergoes such a bifurcation. In this case, the same definitions hold, provided we use the q-times iterated map $\mathbf{f}^{[q]}(\mathbf{p}_i, \rho)$, $i = 1, 2, \ldots, q$.

We now assume the following skew-symmetric form for $\mathbf{f}(\mathbf{x})$:

$$x_{n+1} = \varphi(x_n), \tag{1}$$

$$y_{n+1} = g(x_n, y_n, \rho), \tag{2}$$

such that the dynamics along the x-direction is independent of that in the y-direction, thus $\varphi(x_n)$ acts as a driver signal on the transversal variable y. The Jacobian derivative of maps such as given by (1) and (2) is lower-triangular, and the eigendirections are just x and y, with eigenvalues ξ_x and ξ_y, respectively. We also assume that, in the x-direction, we have strongly chaotic dynamics, such as $\varphi(x) = 2x \pmod 1$. This results in an invariant chaotic set Λ embedded in the two-dimensional phase space.

A particular case of importance is when $g(x, y, \rho)$ has only odd powers in y. In this case, due to the $y \to -y$ symmetry, the line $y = 0$ is an invariant subspace for the system, and the chaotic invariant set Λ is embedded in this invariant subspace. This one-dimensional subspace can be thought of as the synchronization manifold of two suitably coupled one-dimensional maps, after a suitable rotation of axes [12]. There is an infinite number of unstable periodic orbits (UPO) embedded in the chaotic set Λ. In the x-direction, all UPOs will be unstable by construction, i.e., $|\xi_x| > 1$, since the map $\varphi(x)$ is supposed to generate strongly chaotic dynamics for all values of interest of the parameter ρ. The invariant set Λ as a whole can be transversely stable or unstable, and is named a chaotic attractor or a chaotic saddle, respectively, depending on the transversal stability (along the y-direction) of the UPOs embedded in Λ: $|\xi_y| < 1$ (> 1) for the saddle (repeller), with unstable dimension $d^u = 1$ (2).

For maps of the form (1) and (2), UDV occurs as a result of a codimension-one bifurcation acting on the transversal dynamics to Λ, transforming a transversely stable UPO into an unstable one or *vice-versa*. The onset of UDV depends on the value which the eigenvalue ξ_y takes on at the bifurcation point $\rho = \rho_0$. Let $\mathbf{p} = (x = \chi, y = y^*)$ be an unstable fixed point embedded in Λ, and which undergoes such a bifurcation. When $\mathbf{p}$ becomes a repeller, all its infinite preimages also do so. This generates a Lebesgue measure zero set of repellers embedded in the chaotic set Λ, and densely intertwined with a positive measure set of saddles. Trajectories near these newborn repellers may fall into tongues anchored at these points and be repelled away from the chaotic set (Fig. 1). After the onset of UDV, other unstable periodic orbits bifurcate likewise, increasing the number of repellers embedded in the chaotic set and turning the effect of UDV more

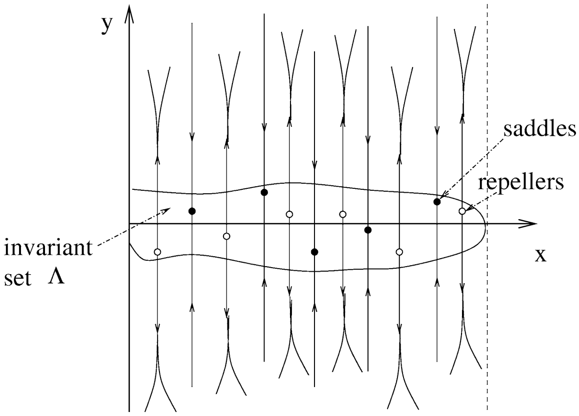

Fig. 1. Schematical view of the intertwined sets of saddles and repellers in the invariant set.

pronounced, as the bifurcation parameter is further increased [18].

A quantitative way to evaluate the local average rates of attraction or repulsion in the transversal dynamics is to compute the finite-time Lyapunov exponents in the y-direction:

$$\lambda_y(x_0, y_0; n) = \frac{1}{n} \sum_{i=1}^{n} \ln \left| \frac{\partial g(x_i, y_i, \rho)}{\partial y_i} \right|. \tag{3}$$

These exponents are typically characterized by a distribution $P(\lambda_y(n))$, such that $P(\lambda_y(n))\, d\lambda_y(n)$ is the relative number of transversal time-n exponents between λ_y and $\lambda_y + d\lambda_y$ [19]. For n large enough, these distributions are Gaussian-like [20], but other distributions have been found to better fit the numerical results [21]. After the onset of UDV, it has been observed that this distribution starts to develop a positive tail, which drifts towards positive values as the UDV is more pronounced [9]. When UDV is the most intense, half of the finite-time transversal exponents are positive, meaning equal contributions of repellers and saddles, such that the average time-n exponent

$$\langle \lambda_y(n) \rangle = \frac{\int_{-\infty}^{+\infty} \lambda_y(n) P(\lambda_y(n))\, d\lambda_y(n)}{\int_{-\infty}^{+\infty} P(\lambda_y(n))\, d\lambda_y(n)} \tag{4}$$

vanishes [18]. In this case, it follows that the infinite-time exponent in the transversal direction, $\lambda_T = \lim_{n\to\infty} \lambda_y(n)$, also vanishes, and the chaotic set Λ loses transversal stability through a blowout bifurcation [22].

The relation between unstable dimension variability and loss of transversal stability can be fully appreciated in the context of the contribution that unstable periodic orbits have on the natural measure of the chaotic invariant set Λ. To compute the infinite-time exponent λ_T, we use typical trajectories on Λ, with respect to its natural measure $\mathsf{m}(\Lambda)$. Since there are an infinite number of unstable periodic orbits embedded in Λ, they are the support of the measure in the sense that, when computing λ_T, such orbits contribute with different weights. These weights, on the other hand, are determined by the magnitude of the unstable eigenvalues of those unstable orbits; such that, the larger is the unstable eigenvalue of the periodic orbit, the smaller is its weight, or contribution to the natural measure. Summing over all unstable period-q orbits embedded in Λ gives then its natural measure [23]

$$\mathsf{m}(\Lambda) = \lim_{q\to\infty} \sum_{\mathbf{p}_q(j)\in\Lambda} \frac{1}{L_u(\mathbf{p}_q(j))}, \tag{5}$$

where $\mathbf{p}_q(j)$ is the jth fixed point of $\mathbf{f}^q(\mathbf{p})$, i.e., $\mathbf{p}_q(j)$ is on a period-r orbit, where r is equal to q or a prime factor of q, and L_u is the expanding eigenvalue of this orbit.

The natural measure associated with the jth period-q orbit is the normalized ratio [18]

$$\mathsf{m}_q(j) = \frac{1/L_u(\mathbf{p}_q(j), q)}{\sum_{\ell=1}^{N_q} [1/L_u(\mathbf{p}_q(\ell))]}, \tag{6}$$

where N_q is the number of period-q orbits. N_q^s and N_q^u are the numbers of transversely stable and unstable period-q orbits, respectively, such that $N_q^s + N_q^u = N_q$. For two-dimensional maps ($q = 1$), N_1^s and N_1^u are the number of saddles and repellers, respectively. The weights of the transversely stable and unstable period-q orbits are given, respectively, by

$$w_q^s = \sum_{j=1}^{N_q^s} \mathsf{m}_q(j) \lambda_2(\mathbf{p}_q(j), q) \quad (\text{for } \lambda_y(\mathbf{p}_q(j), q) < 0), \tag{7}$$

$$w_q^u = \sum_{j=1}^{N_q^u} \mathsf{m}_q(j) \lambda_y(\mathbf{p}_q(j), q) \quad (\text{for } \lambda_y(\mathbf{p}_q(j), q) > 0), \tag{8}$$

where $\lambda_y(\mathbf{p}_q(j), q)$ is time-q transversal Lyapunov exponent for the jth period-q orbit. If $\lambda_y(\mathbf{p}_q(j), q)$

is positive (negative) the periodic orbit is transversely unstable (stable).

Based on these arguments, we can assign the onset of unstable dimension variability of the invariant set Λ to the parameter value ρ for which the first periodic orbit embedded in Λ loses transversal stability. As ρ increases past this critical value, more and more unstable orbits lose transversal stability, and the repellers weight increases with respect to the saddles weight. When $\lambda_T = 0$ the contributions of the saddles and repellers become exactly counterbalanced, and unstable dimension variability is expected to be most intense. At this point, the set Λ loses transversal stability as a whole. As the parameter ρ is further increased, the repellers weight becomes larger than the saddles weight.

The possible types of codimension-one bifurcations of two-dimensional maps can be described, using the center manifold theory, by the normal forms along the transversal direction computed at $\mathbf{p} = (\chi, y^*)$, and written as $z \mapsto g(z, \mu)$, where $z \equiv y - y^*$ and $\mu = \rho - \rho_0$. Hence $g(0, 0) = 0$, by construction. In the following, we treat the possible cases according to the corresponding bifurcation eigenvalue.

Bifurcations with eigenvalue $+1$

In this case $\partial g(z, \mu)/\partial z = 1$ at the bifurcation point $(0, 0)$, for which there are three possibilities.

Pitchfork bifurcation

The normal form describing the dynamics transversal to Λ, at $x = \chi$, is [15]

$$g(z, \mu) = z + \mu z \mp z^3, \tag{9}$$

where the minus sign in the cubic term refers to the supercritical pitchfork bifurcation. There is an invariant subspace at $y = z = 0$, where the chaotic invariant set Λ lies. For $\mu < 0$ the fixed point of the map (9) at $y = 0$ is transversely stable (a saddle), and the onset of UDV is caused by its conversion into a transversely unstable point (a repeller), with the consequent appearance of two saddles outside the chaotic invariant set Λ. The plus sign in (9) is for the sub-critical pitchfork bifurcation, where two repellers outside Λ approach the saddle at $z = 0$ as μ tends to 0 and eventually coalesce there, making the former saddle to become a repeller, as well all its preimages.

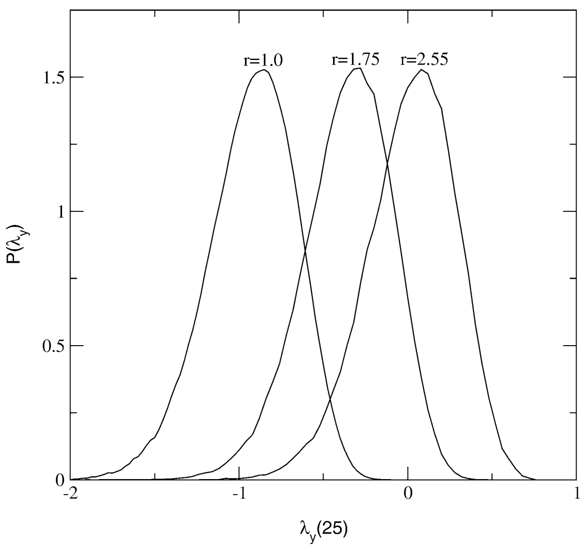

Fig. 2. Distribution of the transversal time-25 Lyapunov exponent for the map (11) and (12) with $a = 4$, $b = 5$, and three values of the bifurcation parameter r.

In both cases the bifurcation diagram presents two curves of fixed points passing through the bifurcation point $(0, 0)$: one curve (the straight line) exists on both sides of the $\mu = 0$ line, whereas the other one lies locally just in one side. Accordingly, the additional conditions for a pitchfork bifurcation to occur at $(0, 0)$ are [15]:

$$\left.\frac{\partial g(z,\mu)}{\partial \mu}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial^2 g(z,\mu)}{\partial z^2}\right|_{(0,0)} = 0,$$
$$\left.\frac{\partial^2 g(z,\mu)}{\partial z \partial \mu}\right|_{(0,0)} \neq 0, \qquad \left.\frac{\partial^3 g(z,\mu)}{\partial z^3}\right|_{(0,0)} \neq 0. \tag{10}$$

The sub-critical pitchfork bifurcation is the mechanism whereby UDV occurs in the riddling map [12, 24]:

$$x_{n+1} = \varphi(x_n) = a x_n (1 - x_n), \quad x \in [0, 1], \tag{11}$$
$$y_{n+1} = r e^{-b(x_n - \chi)^2} y_n + y_n^3, \tag{12}$$

where a is chosen so that the dynamics is chaotic in the invariant subspace $y = 0$, $\chi = 1 - (1/a) = 0.75$ is an unstable fixed point embedded in Λ, and $b > 0$ is kept fixed. For $x = \chi$, Eq. (12) reduces to the normal form (9) when $\mu = r - 1$. The onset of UDV occurs at $r = 1$ and, as r increases past this value, the distributions of the transversal finite-time Lyapunov exponent (Fig. 2) drift toward positive values of λ_y. Note also that the chaotic attractor at $y = 0$ loses transversal stability as a whole at $r^* \approx 2.55$, or a *blowout instability* [25]. We remark that, after the occurrence of UDV, the invariant set is a chaotic saddle.

Transcritical bifurcation

The normal form for the transversal dynamics at the bifurcation point $(0, 0)$ is now

$$g(z, \mu) = z + \mu z \mp z^2. \tag{13}$$

Due to the quadratic term in z, although the $y = 0$ line continues to be an invariant subspace, the chaotic set is not necessarily embedded in it. For $\mu < 0$ (> 0) the fixed point at $z = 0$ is stable (unstable), whereas the other fixed point at $z \neq 0$ is unstable (stable) when $\mu < 0$ (> 0). The minus (plus) sign in (13) refers to a supercritical (sub-critical) bifurcation at $\mu = 0$.

In the bifurcation diagram, or the z–μ plane, there are two curves of fixed points passing through the origin and existing in both sides of the $\mu = 0$ line. Hence, besides the usual conditions $g(0, 0) = 0$ and $\partial g/\partial z(0, 0) = 1$, the following conditions must hold:

$$\left.\frac{\partial g(z,\mu)}{\partial \mu}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial^2 g(z,\mu)}{\partial z^2}\right|_{(0,0)} \neq 0,$$
$$\left.\frac{\partial^2 g(z,\mu)}{\partial z \partial \mu}\right|_{(0,0)} \neq 0. \tag{14}$$

An example of this kind of transition is the following two-dimensional map on the topological cylinder $S^1 \times R^1$

$$x_{n+1} = 2x_n, \quad x \in [0, 2\pi), \tag{15}$$

$$y_{n+1} = \left(y_n + \mu y_n - y_n^2\right) \cos x_n, \tag{16}$$

where $\chi = 0$ is the unstable fixed point embedded in the circle $y = 0$. The y-map (16) reduces to the normal form (13) when $x = \chi$, $(0, 0)$ being the bifurcation point. The onset of UDV at $\mu = 0$ can be seen in the distributions of the time-15 transversal exponent (Fig. 3). Note also that the chaotic attractor loses transversal stability at $\mu \approx 1.0$.

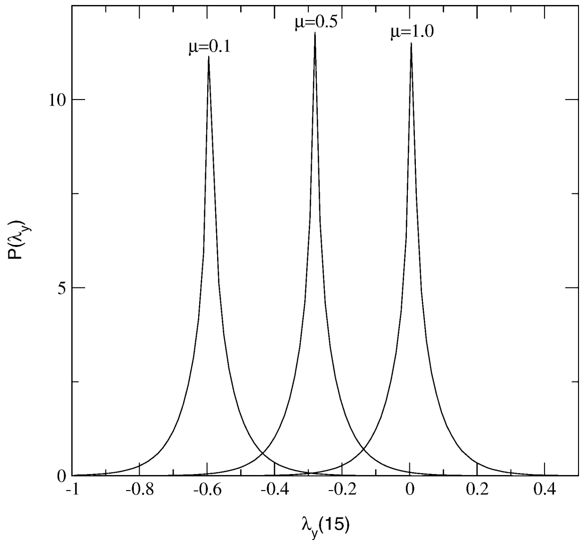

Fig. 3. Distributions of the transversal time-15 Lyapunov exponent for the map (15) and (16) and three values of the bifurcation parameter μ.

Saddle-node bifurcation

The transversal dynamics at the bifurcation point $(0, 0)$ is governed in this case by the normal form

$$g(z, \mu) = z + \mu \mp z^2. \tag{17}$$

For $\mu < 0$ (> 0) there is no fixed point at all and, at $\mu = 0$, a pair of fixed points, one stable and one unstable, appear for $\mu > 0$ (< 0). In the z–μ plane, there is a single curve of fixed points passing through the bifurcation point, which locally lies in the right (the minus sign in (17)) or in the left (the plus sign in (17)) side of the $z = 0$ axis. In either case, the additional conditions for a saddle-node bifurcation are

$$\left.\frac{\partial g(z,\mu)}{\partial \mu}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial^2 g(z,\mu)}{\partial z^2}\right|_{(0,0)} \neq 0. \tag{18}$$

In the saddle-node case, however, it must be observed that UDV occurs rigorously only at the bifurcation point, since before (minus sign) or after (plus sign) this point there is no fixed point (one can say that it is an atypical case of UDV). Accordingly, this scenario was ruled out for the appearance of bubbling, since the bifurcation parameter μ is supposed not to alter the dynamics along the invariant set [26]. When the fixed point $y = y^*$ disappears as a result of a saddle-node bifurcation, the invariant set becomes punctured in a fine scale, with the size of the holes being proportional to μ, and without necessarily a change in the unstable dimension. An example of this case was studied in Ref. [16], where a modified version of the Kaplan–Yorke map was considered. Another example is the non-symmetric version of the riddling map (11) and (12), where a symmetry breaking parameter was added to the transversal map [12].

Bifurcations with eigenvalue -1

The case $(\partial g(z, \mu/\partial z))_{0,0} = -1$ characterizes a *period-doubling bifurcation*, for which the second iterate of the map, $g^{[2]}(z, \mu)$, must undergo a pitchfork bifurcation at the bifurcation point $(0, 0)$. A common choice for the normal form for the transversal dynamics is [15]

$$g(z, \mu) = -z - \mu z + z^3. \tag{19}$$

At $\mu = 0$ the stable fixed point $(z = 0)$ becomes unstable and a stable period-2 orbit emerges out. The conditions for this bifurcation to occur are thus

$$g(0,0) = 0, \qquad \left.\frac{\partial g(z,\mu)}{\partial z}\right|_{(0,0)} = -1, \qquad \left.\frac{\partial g^{[2]}(z,\mu)}{\partial \mu}\right|_{(0,0)} = 0, \tag{20}$$

$$\left.\frac{\partial^2 g^{[2]}(z,\mu)}{\partial z^2}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial^2 g^{[2]}(z,\mu)}{\partial z \partial \mu}\right|_{(0,0)} \neq 0, \qquad \left.\frac{\partial^3 g^{[2]}(z,\mu)}{\partial z^3}\right|_{(0,0)} \neq 0. \tag{21}$$

Like in the pitchfork bifurcation, this case is also characterized by a symmetric normal form in the transversal direction, and $y = 0$ is an invariant subspace containing the chaotic set Λ. An example in $\mathsf{S}^1 \times \mathsf{R}^1$ is

$$x_{n+1} = 2x_n, \quad x \in [0, 1], \tag{22}$$

$$y_{n+1} = \left(-y_n - \mu y_n + y_n^3\right) \cos x_n, \tag{23}$$

with $(\chi = 0, y = 0)$ as the embedded unstable fixed point which loses transversal stability at the onset of UDV, occurring at $\mu = 0$. This is confirmed by the statistics of transversal time-n exponents, the resulting distributions being almost identical to those depicted in Fig. 3. The blowout bifurcation, which signals the loss of the transversal stability of Λ, occurs at $\mu \approx 1$.

In conclusion, we have presented in this Letter the possible scenarios for the onset of UDV in two-dimensional maps, when it occurs due to a codimension-one bifurcation of a fixed point or, possibly, a periodic orbit. We present examples of all the corresponding bifurcation types, classified according to the nature and the sign of the corresponding eigenvalues. The occurrence of UDV, in the examples given, can also be numerically reflected by the fluctuating behavior about zero of the transversal finite-time Lyapunov exponent. Our investigation was restricted to codimension-one bifurcation of maps, but similar bifurcations for vector fields present an analogous classification, *mutatis mutandis*. Bifurcations of higher codimension may also cause the onset of UDV, especially in complex systems, like coupled maps or oscillator lattices, where this fact has already been numerically established [10]. Further research is thus needed to provide a more comprehensive classification of the possible scenarios for the onset of unstable dimension variability.

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