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Mode locking in small-world networks of coupled circle maps

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Abstract

We investigate the emergence of mode locking, or frequency synchronization, in a chain of coupled sine-circle maps with randomly distributed parameters, and exhibiting the small-world property. The coupling prescription we adopt considers the nearest and next-to-the-nearest neighbors of a given site, as well as randomly chosen non-local shortcuts, according to a given probability. A transition between synchronized and non-synchronized patterns is observed as this probability is varied. We also study the statistics of the synchronization plateaus, evidencing a Poisson-type distribution.

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1. Introduction

Condensed-matter physics offers a large number of examples of regular lattices with local interactions between neighbors, for they are the backbones of the crystal structures. On the other hand, there is an increasing interest in lattices which present, in addition to the local interactions, also a small number of non-local couplings, randomly distributed over the lattice. Social networks, for example, are thought to belong to this

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latter category, as illustrated by the famous small-world (or six-degree-of-separation) problem [1,2]. Non-local interactions are also present in lattices of technological (electric power grids), biological (neural networks), and epidemiological (spreading of infectious diseases) interest [3].

It would seem at first that purely random networks could cope with the non-locality problem, but they are not suitable as well, since they do not display the so-called clustering property. Clustering, roughly speaking, is the degree of overlap between neighborhoods (in terms of interactions) of different sites. Regular lattices display a relatively large amount of clustering, but they fail to provide non-local interactions, what accounts for a large average distance between pairs of sites. On the other hand, random graphs have a substantially smaller values of such average distance, which suggests that small-world networks are in between these two limiting situations.

In fact, a small-world network has typically an average distance between sites comparable to the value it would take on a purely random network, while retaining an appreciable degree of clustering, as in regular networks. A realization of this concept was first proposed by Watts and Strogatz [4], who introduced a small amount of randomness in a regular one-dimensional network with periodic boundary conditions. This was done by re-wiring a small fraction of the local connections to new sites randomly chosen along the lattice, so creating the shortcuts necessary to lessen the average distance between sites. This procedure was proved to maintain a high amount of clustering, but it can also generate regions disconnected from the rest of the lattice, what would introduce divergent contributions to the average distance between sites.

An alternative procedure was proposed by Newman and Watts [5], who inserted randomly chosen shortcuts in a regular lattice, instead of re-wiring local interactions into non-local ones. While the clustering properties of the Newman–Watts lattices are similar to the Watts–Strogatz ones, the above-mentioned problem of disconnectedness and infinite distances between sites is circumvented. Since then, the Newman–Watts lattices have been extensively used in a variety of theoretical and numerical studies on small-world networks [6,7].

One of the collective phenomena that have been treated with more attention is the synchronization of periodic or chaotic maps and oscillators coupled in small-world lattices. The fact that small-world lattices facilitate synchronization of coupled phase oscillators has been recognized earlier in the literature [3], and can be understood as an effect of the non-local interactions, provided their relative number is large enough. The synchronization of chaotic systems coupled in small-world lattices was shown to be possible even in the thermodynamical limit [8]. Recently, it has been numerically shown that a small-world network will synchronize for any given coupling strength and a sufficiently large number of sites, even if the original purely regular network cannot achieve synchronization under the same conditions [9].

In most cases investigated so far, the lattice was supposed to be homogeneous, i.e., the sites represented identical dynamical systems (maps or continuous-time flows). There is considerable interest in heterogeneous lattices, for which the systems are non-identical, for example, an assembly of phase oscillators with different normal modes, randomly chosen within a given frequency interval. This type of systems can model an array of lattice-coupled Josephson junctions, for which the microscopic parameters are subjected to fluctuations [10]. Large arrays of Josephson junctions are commonly used as electrical standards. The mode locking, or frequency synchronization of such a system is a problem of practical importance, for the output power delivered by such an array increases as each junction oscillates in a synchronous fashion [11].

In this paper we investigate the frequency synchronization of a heterogeneous small-world network of coupled sine-circle maps, whose properties have been extensively studied as prototypes of perturbed phase oscillators. Each oscillator is endowed with a different normal mode (natural) frequency, randomly chosen from a given interval. For a regular lattice of such systems, with non-local interactions whose strength decreases as a power-law function of the lattice distance, it was numerically shown the existence of a transition between synchronized and non-synchronized states, as the coupling parameters are varied [12,13]. This analysis was further extended to chains of Van der Pol oscillators with the same kind of power-law interaction [14]. Here we consider a similar situation, in the context of the synchronization properties of a small-world network. At zero probability of shortcuts, there was recently reported a first-order transition, for which the normalized shortest-path distance undergoes a discontinuity in the thermodynamical limit [15].

This paper is organized as follows: in the second section we present the coupled map lattice model to be used, and the prescription to introduce the small-world property. Section 3 discusses the emergence of mode locking by using numerical diagnostics of frequency synchronization, as well as the statistical distribution of the synchronization plateaus. Our conclusions are left to the final section.

2. Small-world lattice of coupled circle maps

The normalized phase $\theta(t) \in [0, 1)$ of a continuous-time oscillator, as a limit-cycle system, can be defined in a variety of ways, even when the oscillations amplitudes are behaving chaotically [16]. Periodically forced oscillators can be described by phases, θ_n, sampled at discrete times $n = 0, \pm T, \pm 2T, \pm 3T, \ldots$, where T is the period of the external excitation. This approach has been successfully used to investigate biological clocks described by kicked oscillators [17]. On the other hand, biological generators of rhythms are usually complex assemblies of oscillating units, and their collective behavior is thought to be a desirable operational condition, like in the pacemaker cells which generate the heartbeat [18]. As another example, Josephson junctions have a dynamics well-described by forced pendula with many points of contact with kicked limit-cycle oscillators [10].

Let us consider a one-dimensional lattice of periodically forced oscillators, each of them described by a discrete state variable $\theta_n^{(i)}$ assigned to a site $i = 1, 2, \ldots, N$, with the following coupling prescription [2]: (i) the local part of the interaction considers the nearest and the next-to-the-nearest neighbors of a given site; (ii) there are κ non-local shortcuts randomly chosen along the lattice (Fig. 1). The latter are represented by a non-symmetric connection matrix I_{ij}, whose entries are 0 and 1, randomly chosen according to a probability $p = \kappa/N$. We exclude non-local interactions of a site with itself. Hence, for a lattice with N sites one has $N(N-4)$ matrix elements to be specified.

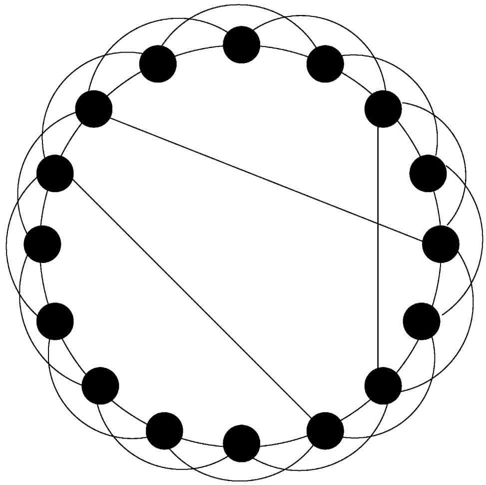

Fig. 1. Schematic representation of a small-world lattice of the type defined by Eq. (1).

A coupled map lattice model consistent with this prescription is

$$\theta_{n+1}^{(i)} = (1-\varepsilon) f(\theta_n^{(i)}) + \frac{\varepsilon}{4+\kappa}\left[f(\theta_n^{(i-1)}) + f(\theta_n^{(i-2)}) + f(\theta_n^{(i+1)}) + f(\theta_n^{(i+2)}) + \sum_{j=1}^{\kappa} f(\theta_n^{(j)}) I_{ij} \right], \quad (1)$$

where $\varepsilon > 0$ is the coupling strength, and the local dynamics at each site is represented by a non-linear circle map $\theta \mapsto f(\theta)$, where $f : [0,1) \to [0,1)$.

The sine-circle map

$$f(\theta) = \theta + \Omega - \frac{K}{2\pi}\sin(2\pi\theta) \pmod 1, \quad (2)$$

provides a simple model for describing the dynamics of a phase oscillator perturbed by a time-periodic forcing, where the parameter $K > 0$ is related to the external forcing amplitude, and $0 \leqslant \Omega < 1$ is the ratio between the natural oscillator frequency and the forcing frequency [19]. We adopt periodic boundary conditions, $\theta_n^{(i)} = \theta_n^{(i \pm N)}$, and use initial conditions $\theta_0^{(i)}$ randomly chosen in the interval $[0, 1)$.

In order to verify that the coupled map lattice built according to Eq. (1) has the properties of a small-world network, we have numerically computed the so-called clustering coefficient C, defined as the average fraction of pairs of neighbors of a site which happen also to be neighbors of each other, and the average separation between sites ℓ in our model. The latter can be analytically estimated by the following expression [7]:

$$\ell(p) = \frac{N}{Z} g(pN(N-4)), \quad (3)$$

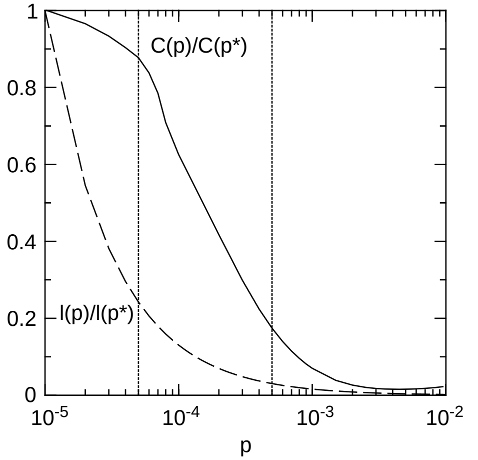

Fig. 2. Normalized clustering coefficient (—) and average separation between sites (- - -) versus probability of non-local connections in the lattice given by Eq. (1), with $N = 10^4$, $\varepsilon = 1.0$, and $p^* = 10^{-5}$. The dotted lines indicate the interval for which the small-world property holds.

where

$$g(x) = \frac{1}{2\sqrt{x^2+2x}} \tanh^{-1}\left(\frac{x}{\sqrt{x^2+2x}}\right), \tag{4}$$

in which $pN(N-4)$ is the number of non-local lattice connections, and Z is the number of regular lattice neighbors at each side (we adopt $Z = 2$ in this paper).

Our results for the shortest path and the clustering coefficient are depicted in Fig. 2, as a function of the probability p of non-local connections. In a small-world network the average lattice distance between two sites must be of the same order as for a random graph: $\ell \sim \ell_{\text{random}}$. For a random graph with $N = 10^4$ and $Z = 2$, it turns out that $\ell_{\text{random}} \cong 2.5$. On the other hand, the clustering coefficient of a small-world lattice must be much greater than for a random graph, namely

$$C \gg C_{\text{random}} = \frac{Z}{N} \sim 10^{-4}\ . \tag{5}$$

In virtue of these requirements, Fig. 2 suggest that the small-world property is displayed when the lattice defined by (1) has non-local shortcuts randomly chosen with a probability specified within the following interval: $5 \times 10^{-5} \lesssim p \lesssim 5 \times 10^{-4}$.

The onset of small-world behavior, shown in Fig. 2 to occur at $p_c \sim 5 \times 10^{-4}$, depends on the lattice size. In Fig. 3 we depict this critical value of probability versus the lattice size N, showing a power-law decrease of p_c as N grows, $p_c(N) \sim 1/N$. In the thermodynamical limit, this critical probability vanishes, as already pointed out in Ref. [15].

The dynamics of an isolated sine-circle map is well-known [19]: if the normal mode frequency Ω is irrational the dynamics on the topological circle $[0, 1)$ is quasi-periodic, a phase never returning to its initial value for large times, and densely fills the circle.

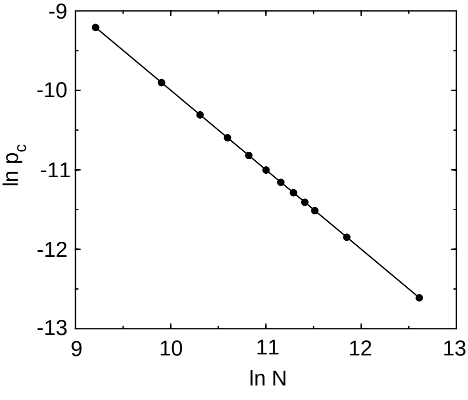

Fig. 3. Critical probability of non-local shortcuts for the onset of small-world behavior in the lattice given by Eq. (1), as a function of the lattice size. The solid line is a least-squares fit with slope −1.0.

When Ω is a rational of the form $\tilde{p}/\tilde{q}$ (with $\tilde{p}$ and $\tilde{q}$ co-prime integers), the motion is periodic (mode locking). If $K = 0$, the set of frequencies for which mode locking occurs has Lebesgue measure zero in the interval $[0, 1)$. As the non-linearity parameter increases, though, mode locking can occur for positive-measure intervals

$$\mathcal{O}_{\tilde{p}/\tilde{q}}(K) \in \left[\frac{\tilde{p}}{\tilde{q}} - \sigma_{\tilde{p}/\tilde{q}}(K)\, , \frac{\tilde{p}}{\tilde{q}} + \sigma_{\tilde{p}/\tilde{q}}(K)\right] \, ,$$

where $\sigma_{\tilde{p}/\tilde{q}}$ increases non-linearly with K, and is generally different for each rational $\tilde{p}/\tilde{q}$.

In the parameter space K versus Ω the mode-locking regions show up as horn-shaped regions (Arnold tongues) anchored at those points, in the $K=0$ line, for which $\Omega = \tilde{p}/\tilde{q}$. The periodic motion within a given Arnold tongue can be characterized by a rational value of the winding number defined as

$$w^{(i)} = \lim_{m\to\infty} \frac{1}{m - n_0} \sum_{n=n_0}^{m} |\theta_{n+1}^{(i)} - \theta_n^{(i)}| \quad (i = 1, 2, \ldots, N) \, , \tag{6}$$

in which n_0 denotes the number of transient iterations we discard.

The winding number is independent of the initial condition $\theta_0^{(i)}$, provided the map $\theta \mapsto f(\theta)$ is invertible. If $K < 1$ the map (2) is invertible, and the unique motions to be expected in this case are periodic and quasi-periodic, the corresponding regions being densely intertwined in the parameter space [19]. When $K > 1$, the map iterations can present chaotic behavior, since $f(\theta)$ ceases to be invertible and the Arnold tongues can overlap. We accordingly limit ourselves to values of $K \leqslant 1$, such that the limit in Eq. (6) is well-defined.

The winding numbers play the role of perturbed frequencies for each forced oscillator and they reduce, for uncoupled maps ($\varepsilon = 0$) and without non-linearity ($K = 0$), to the normal modes $\Omega^{(i)}$ themselves. Non-linearity adds entrainment to the system, since for $K \neq 0$ we can find infinitely many Ω-values belonging to an interval $\mathcal{O}_{\tilde{p}/\tilde{q}}(K)$, i.e., frequencies that are locked to a tongue with a rational winding number. The effect of coupling, for non-linearity well below the threshold of invertibility, is to drive a map to leave its region in parameter space. If a given map jumps out of a given mode-locking region it can fall on a quasi-periodic region through a fold (saddle-node) bifurcation, in which the stable periodic orbit collides with its unstable counterpart, both coalescing at the border of the tongue.

3. Frequency synchronization

Now we consider the dynamics of a lattice of sine-circle maps exhibiting the small-world property, as described by Eq. (1). Complete synchronization between neighbor oscillators would imply the equality of the corresponding phases. For certain applications, however, as in arrays of Josephson junctions, it is sufficient to have equality of the time rates of phase change, hence the interest in analyzing their frequency synchronization.

Two or more circle maps are said to be mode-locked, or synchronized in frequency, if their winding numbers (6) are equal, up to a tolerance, which we set up as 10^{-4}. If the normal mode frequencies $\Omega^{(i)}$ were equal, mode locking would trivially appear as a result of a strong enough coupling. However, it turns out that the natural frequencies $\Omega^{(i)}$ can be different due to various physical reasons. In Josephson junction arrays, for example, this distribution can be ascribed to fluctuations in the microscopic parameters [11]. In order to simulate this distribution we choose $\Omega^{(i)}$ randomly over the interval $[0, 1)$. A diagnostic of frequency synchronization is provided by the winding number dispersion with respect to the lattice average $\bar{w} = (1/N)\sum_{i=1}^{N} w^{(i)}$ or

$$\delta w = \left[\frac{1}{N-1} \sum_{i=1}^{N} (w^{(i)} - \bar{w})^2 \right]^{1/2} , \tag{7}$$

in such a way that low values of δw are related to mode-locked states.

In Fig. 4 we have plotted the average dispersion $\langle \delta w \rangle$ (taken from different initial condition profiles) versus the probability of non-local interactions in a coupled map lattice of form (1). For p close to zero (corresponding to a lattice with essentially only local interactions) this variance is about 0.25, indicating the absence of frequency synchronization. As p builds up inside the small-world interval, this dispersion decreases to ≈ 0.05, this being practically the saturation value for higher probabilities. The decrease of the winding number dispersion within the small-world interval is best shown in a log–log plot (see Fig. 5), where the linear fit suggests a power-law decay $\langle \delta w \rangle \sim p^{-\gamma}$, where γ is close to unity.

Coupling can induce synchronization, depending on the balance between diffusion and disorder in the lattice. For a regular lattice with nearest-neighbor interactions only, we have observed a pronounced tendency to avoid synchronization, in opposition to regular lattices with global ("mean-field") couplings. When the coupling strength decreases with the lattice distance as a power-law, we have found a transition between synchronized and non-synchronized states, as the effective range varies from large to small values, corresponding to global and local couplings, respectively [12–14]. The

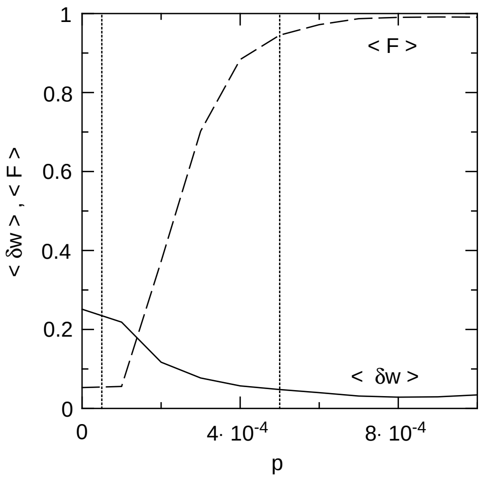

Fig. 4. Winding number dispersion (—) and frequency order parameter (- - -) versus probability of non-local connections in the lattice given by Eq. (1), with $N = 10^4$ and $\varepsilon = 1.0$. The dotted lines indicate the interval for which the small-world property holds.

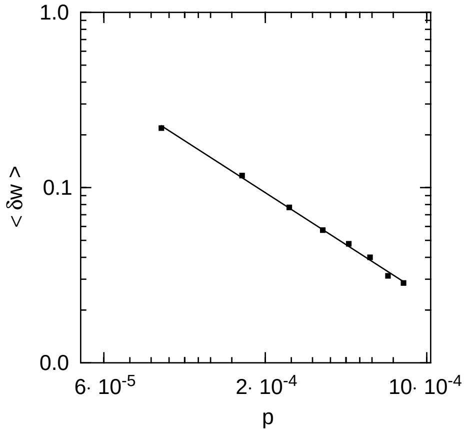

Fig. 5. Decay of the average winding number dispersion with probability of non-local connections, in a small-world lattice with $N = 10^4$ and $\varepsilon = 1.0$. The solid line is a least-squares fit with slope -0.98.

case of small-world lattices can be treated in the same spirit, by considering the probability of non-local interactions as a tunable parameter.

We also computed, as another diagnostic of synchronization, the so-called frequency order parameter [20] $F = N(\bar{w})/N$, where $N(\bar{w})$ is the number of sites with winding number equal to the space average $\bar{w} \approx 0.5$, the latter value resulting from the assumed uniform distribution of natural frequencies over the $[0, 1)$ interval. In a completely non-synchronized lattice there would be a small number of sites with winding numbers equal to 0.5, in a fraction no greater than any other particular value of w, hence

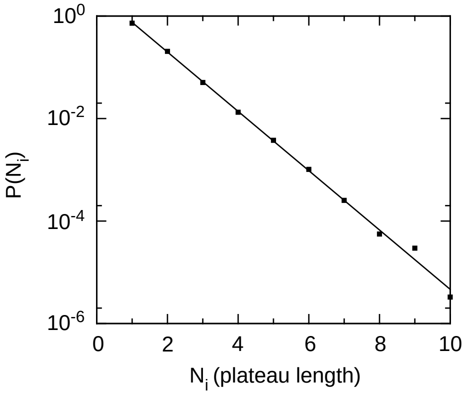

Fig. 6. Histogram of synchronization plateau lengths for a small-world lattice with $N = 10^5$, $\varepsilon = 0.05$, and $p = 10^{-4}$. The solid line is a least-squares fit with slope -1.3327.

$F \sim (1/N) \approx 0$ for large N. For a completely synchronized lattice, on the contrary, all sites share this winding number, so that $N(\bar{w}) = N$ or $F = 1$. Our results for a small-world lattice, in which the average order parameter $\langle F \rangle$ was taken by considering different initial conditions for the lattice, are also displayed in Fig. 4, where the increase of $\langle F \rangle$ within the small-world interval confirms the tendency of maps to lock their frequencies as a result of a small number of shortcuts added to an otherwise regular lattice.

A synchronization plateau is defined as a cluster of neighbor sites for which the maps have the same winding numbers. We have typically a statistical distribution of the sizes of these plateaus, as can be observed in Fig. 6, where we present a histogram for the plateau lengths N_i, $i = 1, 2, \ldots, N_p$, showing that small plateaus are far more common than large ones, as it has previously shown to occur for short-range couplings of circle maps [13]. The least-squares fit in Fig. 6 suggests an exponential distribution $P(N_i) = a \exp(-bN_i)$, which indicates a small average plateau length $\langle N_i \rangle = 1/b \approx 0.75$. Other choices of p would give similar results for the distribution, provided we remain in the small-world regime. The value of the distribution slope b, however, changes with the coupling strength ε, as evidenced by Fig. 7. The slope initially decreases as the coupling becomes stronger, and after a coupling strength of about 0.05 it increases again. For even higher ε-values, there is no exponential fitting to the numerical data. The minimum value displayed by the distribution slope corresponds to a maximum average plateau size, with respect to the coupling strengths here investigated.

4. Conclusions

We have observed that the small-world property facilitates mode locking or frequency synchronization of coupled oscillators with natural frequencies randomly distributed over a specified interval. It should be noted that such a system would never exhibit mode locking if only local interactions (nearest and next-to-the-nearest neighbors) were considered.

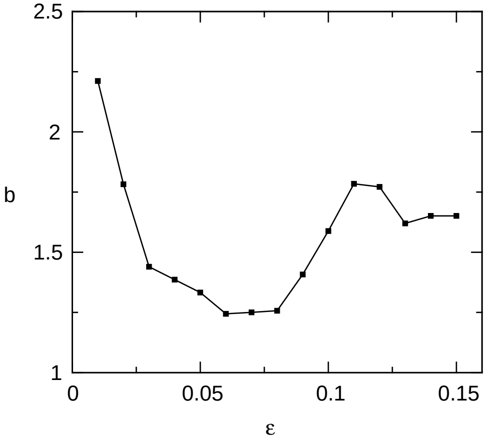

Fig. 7. Variation of the slope of the synchronization plateau distribution with the coupling strength, for a small-world lattice with $N = 10^5$ and $p = 10^{-4}$.

Synchronization is then the outcome of a competition between two antagonistic factors: (i) the frozen disorder represented by the random distribution of normal mode frequencies and (ii) the diffusive effect of coupling, spreading the interaction effect along the lattice. When the latter factor overcomes the former, there will be frequency synchronization. This explains why small-world lattices are so effective in achieving synchronization, since the shortcuts randomly introduced in the chain—if the related probability is large enough—provide a means of surpass the frozen disorder which prevents mode locking in purely local lattices. Moreover, the distribution of clusters of synchronized maps shows a pronounced tendency to a large number of small clusters.

We have used a number of numerical diagnostics to come to these results: the winding number dispersion, the frequency order parameter, and the statistics of the synchronization plateaus. Since our results are weakly dependent on the specific form of the circle map used, we argue that similar results would be obtained for general continuous time oscillators, in as much as these maps may also be regarded as stroboscopic plots of time-periodic flows.

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