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# Kolmogorov–Sinai entropy for locally coupled piecewise linear maps

Antônio M. Batista[1], Ricardo L. Viana *

*Departamento de Física, Universidade Federal do Paraná, 81531-990, Curitiba, Paraná, Brazil*

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**Abstract**

We present analytical results for the Kolmogorov–Sinai entropy of a one-dimensional lattice of locally coupled piecewise linear maps, for some particular values of the coupling strength. Our results explain the numerically observed fact that the entropy of a lattice of chaotic maps increases for strong coupling. © 2002 Published by Elsevier Science B.V.

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## 1. Introduction

Coupled map lattices are prototypes of spatially extended dynamical systems, which present discrete space and time, allowing a continuous state variable [1]. They have been intensively used for studies of spatio-temporal phenomena, like pattern formation [2], traveling waves [3], fully developed turbulence [4], and synchronization [5]. One of the reasons for the popularity exhibited by coupled map lattice models is the small CPU time necessary to follow interesting spatio-temporal patterns, compared with chains of coupled oscillators or partial differential equations [6]. The Lyapunov spectrum of a coupled map lattice gives us information about its degree of chaoticity, when one or more Lyapunov exponents are positive.

The Lyapunov spectrum of piecewise linear and logistic maps with local coupling has been previously studied by Kaneko [7], and by Isola et al. [8], who stressed

* Corresponding author.
*E-mail address:* viana@fisica.ufpr.br (R.L. Viana).

[1] Permanent address: Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, 84033-240, Ponta Grossa, Paraná, Brazil.


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its connection with the spectrum of the discrete Schroedinger operator in quantum mechanics [9]. The corresponding spectrum for lattices of logistic maps with global coupling was also considered by Kaneko [10]. In a previous work [11], we numerically obtained the Lyapunov spectrum of a lattice of chaotic logistic maps with a coupling prescription in which the interaction between sites decreases with the lattice distance in a power-law fashion. This coupling scheme reduces to the previously studied cases of local and global couplings respectively, and our results agree with those already observed for these limiting cases [8,10].

Many quantities of interest can be extracted from the Lyapunov spectrum of a coupled map lattice, like the maximal and mean Lyapunov exponents, the Kolmogorov–Sinai (KS) entropy, and the Lyapunov dimension. The KS-entropy is particularly important since it is the asymptotic rate of creation of information by the dynamical system [9], and thus furnishes a quantitative measure of its degree of chaoticity. Moreover, the knowledge of the KS-entropy enables us to apply the thermodynamical formalism, if the coupled map lattice model satisfies a large deviation statistics [12].

The KS-entropy is originally based on the Shannon entropy. For systems with long-range interactions or long-time memory effects, a non-extensive form of entropy introduced by Tsallis [13] has produced a generalized version of the KS-entropy [14]. This form is useful in the description of the Lyapunov spectrum of coupled map lattices with non-local interactions, like a power-law coupling [11,15].

Unfortunately, for most coupled map lattices the Lyapunov spectrum and the corresponding KS-entropy may be determined only numerically. This poses a difficulty for analyzing the behavior of the KS-entropy—and the corresponding thermodynamical quantities which depend on it—under variations of the coupling and dynamical parameters. For piecewise linear maps, however, the constant slope enables us to obtain the Lyapunov spectrum of the corresponding coupled map lattice. This calculation has been done for the local coupling case [7,8]. The purpose of this work is to obtain approximate analytical expressions for the KS-entropy for such a model, in order to explain its behavior with respect to the coupling strength, in particular its increase for strong coupling.

This paper is organized as follows: in Section 2 we present the Lyapunov spectrum of a locally coupled map lattice and study its distinct features with respect to the possible values of the coupling strength. Section 3 presents the KS-entropy and Lyapunov dimension for the system, and Section 4 is devoted to analytical calculations of some particular cases of interest. The last section contains our conclusions.

## 2. Lyapunov spectrum

We consider a lattice of $N$ coupled piecewise linear maps $x \mapsto f(x) = \beta x \,(\mathrm{mod}\, 1)$, where $x_n^{(j)} \in [0,1)$ represents the state variable for the site $j$ $(j = 1, 2, \ldots, N)$ at time $n$, and $\beta > 1$. The local coupling prescription we use connects nearest neighbors only [2,6]:

$$x_{n+1}^{(j)} = (1-\varepsilon) f(x_n^{(j)}) + \frac{\varepsilon}{2}[f(x_n^{(j+1)}) + f(x_n^{(j-1)})]\,, \tag{1}$$

where $0 \leqslant \varepsilon \leqslant 1$, and the uncoupled maps have Lyapunov exponent $\lambda_U = \ln \beta$ for almost all initial conditions $x_0 \in [0,1)$. By "almost all" we mean except a Lebesgue measure zero set of points for which the map is discontinuous: $x_k = 1/\beta^k$, with integer $k$ [16]. For $\beta > 1$ the uncoupled maps typically present chaotic dynamics.

The coupled map lattice (1) is an $N$-dimensional discrete dynamical process: $x_{n+1}^{(j)} = \mathscr{F}^{(j)}(x_n^{(1)}, \ldots, x_n^{(N)})$ $(j = 1, \ldots, N)$, and the corresponding Lyapunov spectrum is formed by $N$ exponents, one for each independent eigendirection $\mathbf{u}_i$ in the tangent space: $\lambda_1 = \lambda_{max} > \lambda_2 > \cdots > \lambda_N$. The sum of all exponents $\lambda_1 + \lambda_2 + \cdots + \lambda_N$ is the time rate of change of volumes in the $N$-dimensional Euclidean phase space [1].

The Lyapunov exponent corresponding to the eigendirection $\mathbf{u}_j$ is

$$\lambda_j = \lim_{n \to \infty} \left( \frac{1}{n} \right) \ln \| \mathbf{A}_n . \mathbf{u}_j \| , \tag{2}$$

where $\mathbf{A}_n = \prod_{\ell=1}^{n} \mathbf{J}_\ell$ is the product of $n$ Jacobian matrices with elements $[\mathbf{J}_\ell]_{ij} = \partial \mathscr{F}^{(i)} / \partial x_\ell^{(j)}$. Using periodic boundary conditions, $x_n^{(0)} = x_n^{(N)}$ and $x_n^{(1)} = x_n^{(N+1)}$, the Jacobian matrices are symmetric and circulant:

$$[J]_{ij} = \beta \begin{pmatrix} 1-\varepsilon & \varepsilon/2 & 0 & \cdots & \varepsilon/2 \\ \varepsilon/2 & 1-\varepsilon & \varepsilon/2 & \cdots & 0 \\ 0 & \varepsilon/2 & 1-\varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon/2 & \cdots & \cdots & \varepsilon/2 & 1-\varepsilon \end{pmatrix} . \tag{3}$$

The eigenvalues of a circulant matrix

$$[C]_{ij} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_{k-1} & c_0 & \cdots & c_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix} \tag{4}$$

are given by the complex roots of a characteristic polynomial [17],

$$P(s_j) \equiv c_0 + c_1 s_j + c_2 s_j^2 + \cdots + c_{k-1} s_j^{k-1} = 0 , \tag{5}$$

in the form $s_j = \exp(2\pi i j/N)$ (in the following $\mathrm{i} = \sqrt{-1}$ unless explicitly stated) [18]. With them, the eigenvalues of the Jacobian matrix (related to the eigendirections $\mathbf{u}_j$) are

$$\sigma_j = \beta \left[ 1 - \varepsilon + \frac{\varepsilon}{2} s_j (1 + s_j^{N-2}) \right] , \tag{6}$$

in such a way that the Lyapunov spectrum for Eq. (1) is [7,8]

$$\lambda_j = \ln \beta + \ln \left| 1 - \varepsilon \left[ 1 - \cos\left( \frac{2\pi j}{N} \right) \right] \right| \quad (j = 1, 2, \ldots, N) . \tag{7}$$

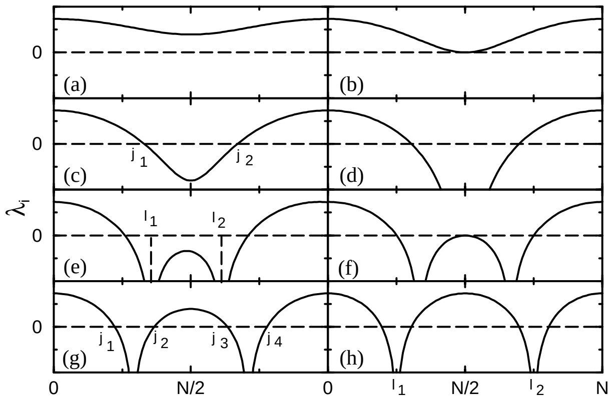


Fig. 1. Lyapunov spectrum for a lattice of piecewise linear maps with coupling strengths: (a) $0 < \varepsilon < \varepsilon_{c1}$; (b) $\varepsilon = \varepsilon_{c1}$; (c) $\varepsilon_{c1} < \varepsilon < \frac{1}{2}$; (d) $\varepsilon = \frac{1}{2}$; (e) $\frac{1}{2} < \varepsilon < \varepsilon_{c2}$; (f) $\varepsilon = \varepsilon_{c2}$; (g) $\varepsilon_{c2} < \varepsilon < 1$; (h) $\varepsilon = 1$.

The maximal Lyapunov exponent is $\lambda_U = \ln \beta$, and it typically decreases with $j$, although the Lyapunov spectrum may assume qualitatively different features according to the values assumed by the coupling strength, which is schematically shown in Fig. 1. In Fig. 1(a) all Lyapunov exponents are positive. Due to the symmetric character of the coupling term (sites to the left- and right-hand side contribute in the same foot), the spectrum is also symmetric with respect to $j = N/2$, where the corresponding Lyapunov exponent is denoted $\lambda_m$. A necessary condition for a spectrum like that depicted in Fig. 1(a) is that $\lambda_m > 0$. Applying this condition to Eq. (7) gives two inequalities

$$\varepsilon < \varepsilon_{c1} \equiv \frac{1}{2}\left(1 - \frac{1}{\beta}\right) \quad \text{or} \quad \varepsilon > \varepsilon_{c2} \equiv \frac{1}{2}\left(1 + \frac{1}{\beta}\right) . \tag{8}$$

For $\beta = 2$ we have $\varepsilon_{c1} = \frac{1}{4}$ and $\varepsilon_{c2} = \frac{1}{4}$. However, the sufficient condition is that all exponents are positive, which rules out the second inequality in (8). Thus, Fig. 1(a) is only typical for $\varepsilon < \varepsilon_{c1}$. In this case $\lambda_m$ is also the minimum value of the Lyapunov exponents, and vanishes for $\varepsilon = \varepsilon_{c1}$ (Fig. 1(b)).

As the coupling strength is further increased (Fig. 1(c)), there are some negative exponents for $j_1 < j < j_2$, where $j_{1,2}$ are roots of expression (7), satisfying

$$j_{1,2}(\varepsilon) = \frac{N}{2\pi} \arccos\left[1 - 2\left(\frac{\varepsilon_{c1}}{\varepsilon}\right)\right] . \tag{9}$$

This situation holds until $\varepsilon = \frac{1}{2}$ (Fig. 1(d)), where the exponent $\lambda_{N/2}$ diverges. For $\frac{1}{2} < \varepsilon < \varepsilon_{c2}$, the typical situation is depicted in Fig. 1(e), where there are two divergent exponents at $j = \ell_{1,2}$, given by

$$\ell_{1,2}(\varepsilon) = \frac{N}{2\pi} \arccos(2\varepsilon_{c1}) . \tag{10}$$

As $\varepsilon = \varepsilon_{c2}$, as given by Eq. (8) (Fig. 1(f)), the exponent $\lambda_{N/2}$ vanishes again, and afterwards more exponents become positive (Fig. 1(g)). The intervals for which the exponents are positive, namely $(0, j_1)$, $(j_2, j_3)$, and $(j_4, N)$, have their limits given by real and positive roots of (7). Finally, for $\varepsilon = 1$ (Fig. 1(h)) the middle exponent $\lambda_{N/2}$ is $\lambda_U$, and the points of divergence are $\ell_1(1) = N/4$ and $\ell_2(1) = 3N/4$, respectively.

## 3. Kolmogorov–Sinai entropy

The maximal Lyapunov exponent $\lambda_{max}$ is the exponential rate at which an arbitrarily small displacement is amplified with time, so it is sufficient that $\lambda_{max} > 0$ for the system to exhibit chaotic dynamics *stricto sensu*. For coupled chaotic maps many exponents are positive, hence a quantity of interest is the average sum of the positive Lyapunov exponents [18]

$$h = \langle \lambda_j \rangle_{j, \lambda_j > 0} = \frac{1}{N} \sum_{j=1}^{\lambda_j > 0} \lambda_j \; . \tag{11}$$

If the dynamical system is a diffeomorphism with an invariant ergodic measure absolutely continuous with respect to the Lebesgue measure, then this quantity is equal to the density of KS-entropy, which is the asymptotic rate of creation of information by successive iterations of the dynamical process (Pesin's theorem) [19]. For $C^1$ maps preserving an ergodic measure, $h$ is an upper bound for the KS-entropy [20]. The equality between $h$ and the density of KS-entropy is generally valid for systems having a Sinai–Ruelle–Bowen (SRB) measure, like Axiom-A systems [9].

The SRB measure is supported on an attractor and describes the statistics of long-time behavior of the orbits, for almost every initial condition in the corresponding basin of attraction, with respect to the Lebesgue measure. SRB measures were constructed for lattices of coupled circle maps [21]. Although there are existence results on SRB measures in a general context [22], as far as we know there is no rigorous proof that such measures exist for lattices of form (1). The technical difficulty lies in the non-existence of Markov partitions for coupled maps (even if they exist for the uncoupled maps).

Fig. 2 shows the numerically obtained KS-entropy for a lattice of $N = 501$ coupled maps with $\beta=2$, random initial conditions $x_0^{(j)}$ and periodic boundary conditions, *versus* the coupling strength. For vanishing coupling ($\varepsilon = 0$) we have simply $h = \lambda_U = \ln 2$ which is the Lyapunov exponent for uncoupled maps. As $\varepsilon$ builds up, the coupling reduces the number of positive exponents (see Figs. 1(a)–(e)), decreasing the entropy to a minimum value $h_{\min} \approx 0.2$. After that, however, the entropy ceases to decrease with the coupling strength and even increases for large coupling. This occurs since for $\varepsilon > \varepsilon_{c2} = \frac{3}{4}$ some exponents become positive due to the absolute value in Eq. (7) (see Figs. 1(f)–(h)).

This fact was already noted by Kaneko [2] in lattices of locally coupled logistic maps at crisis $x \mapsto f(x) = 1 - 2x$. We also obtained this behavior in the short-range limit of a power-law coupling [11]. This “valley” of lower entropies for intermediate coupling is related to some classes of weakly chaotic behavior, or non-fully turbulent spatio-temporal patterns, namely, defect turbulence, Brownian motion of defects, and pattern selection (in increasing order of coupling strength) [1].

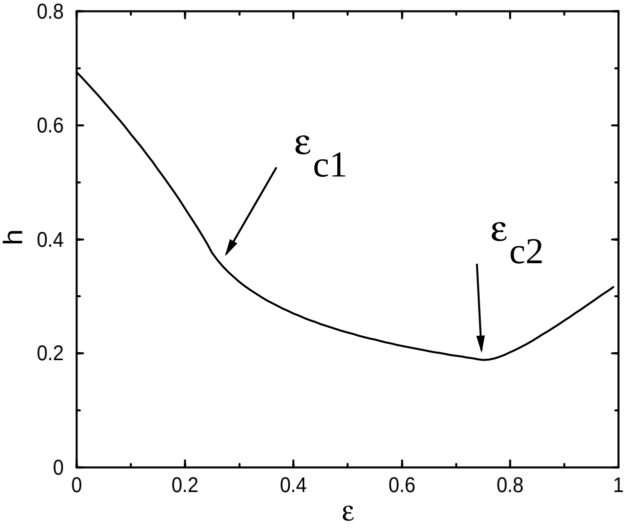


Fig. 2. KS-entropy density in terms of the coupling strength, for a lattice of $N = 501$ piecewise linear maps with $\beta = 2$, with periodic boundary conditions and random initial conditions.

Another quantity of interest is the Lyapunov dimension. Let $p$ be the greatest integer for which $\sum_{j=1}^{p} \lambda_j \geqslant 0$. The Lyapunov dimension is defined as

$$D = \begin{cases} 0 & \text{if there is no such } p\,, \\ p + \dfrac{1}{|\lambda_{p+1}|} \displaystyle\sum_{j=1}^{p} \lambda_j & \text{if } p < N\,, \\ N & \text{if } p = N\,. \end{cases} \tag{12}$$

It is conjectured that $D$ is equal to the information dimension of the system [23]. The properties of the Lyapunov dimension are quite similar to the KS-entropy. In particular, it also decreases for intermediate coupling strengths and then increases back, an effect observed for locally coupled piecewise linear maps by Boldrighini and coworkers [24].

## 4. Analytical results for the KS-entropy

Substituting Lyapunov spectrum (7) into (11) gives the KS-entropy in terms of a summation which, generally, does not have a closed analytical form. Approximate values, however, can be obtained by supposing that the lattice is so tightly packed that the distance between sites $j$ is small enough to allow us to consider it as a continuous value, and to replace the summation by an integral

$$h(\varepsilon) = \frac{1}{N} \int_{\lambda > 0} \mathrm{d}j \left\{ \ln \beta + \ln \left| 1 - \varepsilon \left[ 1 - \cos \left( \frac{2\pi j}{N} \right) \right] \right| \right\} , \tag{13}$$

where the integral runs only over the positive values of $\lambda$. This constraint leads to the integration limits $(j_{min}, j_{max})$, according to the different categories for Lyapunov exponents depicted in Fig. 1.

Performing the change of variable from $j$ to $x = 2\pi j/N$ we have

$$h(\varepsilon) = \frac{1}{2\pi}(x_{max} - x_{min}) \ln \beta + I_\varepsilon(x_{max}, x_{min}) , \tag{14}$$

where

$$I_\varepsilon(x_{max}, x_{min}) = \frac{1}{2\pi} \int_{x_{min}}^{x_{max}} \mathrm{d}x \ln |1 - \varepsilon(1 - \cos x)| . \tag{15}$$

For $0 < \varepsilon \leqslant \varepsilon_{c1}$ all exponents are positive, so the integral runs from $j = 0$ to $N$ (we shifted the origin from $j = 1$ to 0 without appreciable change of results). If $\varepsilon < \frac{1}{2}$ there is no need for the absolute value, and we may write

$$h(\varepsilon) = \ln \beta + I_\varepsilon(0, 2\pi) . \tag{16}$$

We can obtain an approximate analytical value for the integral $I_\varepsilon$ for the case of small coupling ($\varepsilon \ll 1$). Defining $y = \varepsilon(1 - \cos x)$, the fact that $\max y = \varepsilon$ enables us to expand the integrand in (15) in powers of $y$ so that

$$I_\varepsilon(0, 2\pi) = -\frac{1}{2\pi} \sum_{q=1}^{\infty} \frac{\varepsilon^q}{q} \int_0^{2\pi} \mathrm{d}x (1 - \cos x)^q . \tag{17}$$

Since the above infinite series converges uniformly and absolutely for $\varepsilon < 1$, we can obtain an expression in powers of $\varepsilon$ so that the entropy in this case is

$$h(\varepsilon) = \ln \beta - \varepsilon - \tfrac{3}{4}\varepsilon^2 - \tfrac{5}{6}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \quad (0 < \varepsilon < \varepsilon_{c1}) . \tag{18}$$

In Fig. 3 we compare the (exact) numerical result for the KS-entropy density with two approximations obtained from Eq. (18) by retaining second- and third-order terms in $\varepsilon$. We see that the agreement is very good, except near $\varepsilon_{c1}$. After this value ($\varepsilon_{c1} < \varepsilon < \frac{1}{2}$) there are some negative Lyapunov exponents, corresponding to the case depicted in Fig. 1(b). These negative exponents belong to the interval $j_1 < j < j_2$, whose limits are given by (9). Using the symmetry of spectrum (7) we have

$$h(\varepsilon) = \frac{1}{\pi} x_1 \ln \beta + 2 I_\varepsilon(0, x_1) . \tag{19}$$

Retaining terms only up to the quadratic ones in $\varepsilon$, a similar algebra gives

$$I(\varepsilon) = \frac{1}{\pi} x_1 \ln \beta - \varepsilon(x_1 - \sin x_1) - \frac{\varepsilon^2}{2} \left( \frac{3}{2} x_1 - 2 \sin x_1 + \frac{1}{4} \sin 2x_1 \right) + \mathcal{O}(\varepsilon^3) \quad (\varepsilon_{c1} < \varepsilon < \tfrac{1}{2}) . \tag{20}$$

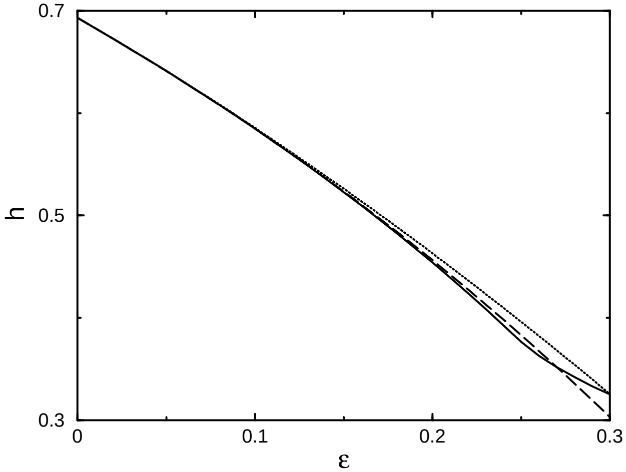


Fig. 3. KS-entropy density versus coupling strength. The analytical results from Eq. (18) are shown with quadratic (dotted line) and cubic terms (dashed line). The solid line represents the numerical result.

However, since for $\beta = 2$ the condition $\frac{1}{4} < \varepsilon < \frac{1}{2}$ implies larger $\varepsilon$-values than in the preceding case, we do not expect a good agreement between numerical and analytical results from Eq. (20) in this case. Accordingly, we shall not present results for the case $\frac{1}{2} < \varepsilon < 1$.

Another case in which integral (15) may be approximately evaluated in a closed form is for $\varepsilon = 1$, which is the largest strength considered in this paper. In this case, the Lyapunov spectrum has the form of Fig. 1(e), where there are four regions of positive exponents with the same positive area, in such a way that the entropy is now

$$h(1) = \frac{2}{\pi}\tilde{x}_1 \ln \beta + 4I_1(0, \tilde{x}_1)\,, \tag{21}$$

where

$$\tilde{x}_1 = x_1(\varepsilon = 1) = \arccos\left(\frac{1}{\beta}\right)\ . \tag{22}$$

The integral

$$2\pi I_1(0, \tilde{x}_1) = \int_0^{\tilde{x}_1} \ln\cos x \, \mathrm{d}x \tag{23}$$

has the following series expansion (that converges provided $\tilde{x}_1 < 1$):

$$-\frac{1}{6}\tilde{x}_1^3 - \frac{1}{60}\tilde{x}_1^5 - \frac{1}{315}\tilde{x}_1^7 - \cdots - \frac{2^{2n-1}(2^{2n}-1)B_n}{n(2n+1)!}\tilde{x}_1^{2n+1} - \cdots\,, \tag{24}$$

where $B_n$ are the Bernoulli numbers [25].

Substituting (24) into Eq. (21) we have the result

$$h(1) = \frac{2\tilde{x}_1}{\pi}\left[\ln\beta - \frac{1}{6}\tilde{x}_1^2 + \frac{1}{60}\tilde{x}_1^4 + \mathcal{O}(\tilde{x}_1^7)\right]\ . \tag{25}$$

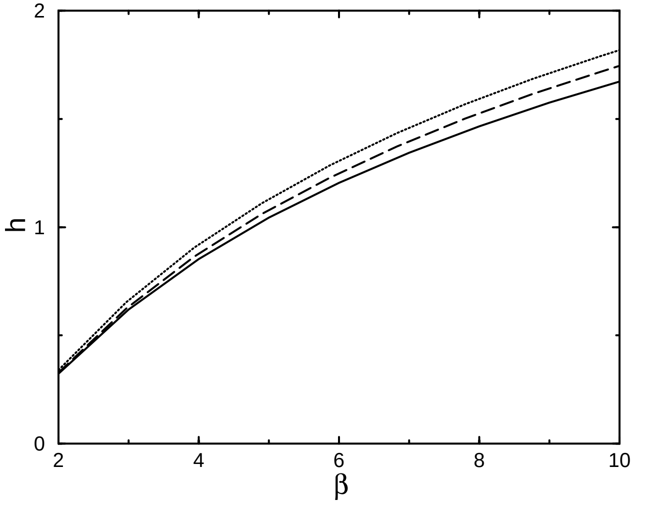


Fig. 4. KS-entropy density for the $\varepsilon = 1$ case versus the slopes of the coupled maps. The analytical results from Eq. (25) are shown with cubic (dotted line) and fifth-order terms (dashed line). The solid line represents the numerical result.

In Fig. 4 we compare the exact numerical result for $\varepsilon = 1$ with two truncated forms of the above approximate formula, for different values of $\beta$. In the case of $\beta = 2$, it turns out that the agreement is already excellent even for the lowest order correction. For large $\beta$, the analytical results are still useful. If $\beta = 10$, for example, the relative error is smaller than 10%.

## 5. Conclusions

For piecewise linear maps the slope is constant, so that the Lyapunov spectrum of a lattice of coupled maps of this kind can be analytically obtained. We have studied the properties of this spectrum with respect to variations of the coupling strength of the lattice. A quantity of interest is the KS-entropy, which is equal to the sum of the positive Lyapunov exponents for a closed system, and measures the rate at which information is being created. For weak coupling the entropy density decreases from its maximum value (equal to the Lyapunov exponent of the uncoupled maps) and achieves a minimum value for intermediate coupling.

This "valley" of smaller chaoticity is related to spatio-temporal patterns characterized by weak chaos, like pattern competition or defect turbulence. As the coupling strength increases from zero, the Lyapunov spectrum is pushed downwards, and after a critical value, some exponents become negative and are ruled out in the computation of the entropy, which decreases as a consequence. However, for a further increase of the coupling strength, some of these negative exponents become progressively positive. The net result is an entropy increase, but at a smaller rate. In addition, we have obtained analytical approximate results for the entropy, both for weak and strong coupling, that are in agreement with numerical results. Although the calculations were performed for piecewise linear maps, this behavior for the entropy with respect to the variation of the coupling strength is fairly typical for coupled chaotic maps.

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