Synchronization plateaus in a lattice of coupled sine-circle maps

Sandro E. de S. Pinto and Ricardo L. Viana
Departamento de Física, Universidade Federal do Paraná, 81531-990 Curitiba, Paraná, Brazil
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Frequency synchronization is a common phenomenon in spatially extended dynamical systems, like oscillator chains and coupled map lattices. We study the distribution of synchronization plateaus in a sine-circle map lattice with a variable range coupling and randomly distributed natural frequencies. A transition between synchronized and nonsynchronized phases is observed as the coupling range is varied. The lengths of the synchronization plateaus are found to obey an exponential distribution for local coupling.

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I. INTRODUCTION

When a physical oscillator dynamical state depends not only on its own equation of motion but also on the state of its neighbors, these oscillators may synchronize, in the sense that their oscillation periods become equal. This fact has been first described by Huyghens, who studied it for two mechanically coupled pendula. Systems of coupled oscillators occur in many physical and biological contexts, as they may describe or model Josephson junction arrays [1], laser arrays, and cardiac and neuronal rhythms [2]. In this perspective, it would be important to characterize how the ensemble of oscillators passes from a nonsynchronized to a completely synchronized state, since the latter indicates the emergence of coherent behavior. As a collective effect, one may expect that even chaotic oscillators embedded in the ensemble could become periodic by means of the mutual interaction. In this sense, we may regard the synchronized state as a kind of spatiotemporal ''attractor'' [3].

If the interacting oscillator units are described by systems of ordinary differential equations, there are many theoretical results about the collective dynamics of such an ensemble, in particular its synchronization properties [4]. But in many cases the available oscillator model is a discrete-time map. Circle maps can be used when the dynamical variable of interest is a normalized angular phase (the circle is mapped into itself). In many cases it is possible to describe a limit-cycle oscillator with periodic impulsive excitation by using a circle map, in the limit of very large dissipation [5]. In some continuous time oscillators, like the Rössler system, it is possible to define such a phase for synchronization studies [6].

Chains of continuous-time coupled oscillators have been used for investigating spatially extended dynamical systems since the seminal work of Fermi, Pasta and Ulam about energy equipartition in nonlinear lattices [7]. Models of this kind are particularly important in studies involving arrays of coupled Josephson junctions [1]. One problem about their use, however, consists of the large computer time necessary to yield reliable results, even when parallel computing is employed. One would benefit from a simpler model from the computational point of view, yet retaining some of the most important dynamical features of the physical problem.

Lattices of coupled maps have been studied for more than one decade as models of spatiotemporal phenomena [8]. They are dynamical systems with discrete space and time, but with a continuous state variable, or amplitude, for each lattice site. In the case of an array of circle maps, we may assign an oscillator phase as the state variable for each site, forming a chain of coupled dynamical units. As long as a map can provide a simplified model of a perturbed nonlinear oscillator, coupled circle maps present features similar to chains of continuous-time oscillators. One of these phenomena turns to be the capability of synchronization. The spatiotemporal behavior is governed by two simultaneous mechanisms: the intrinsic nonlinear dynamics of each map, and diffusion due to the spatial coupling between maps.

The most widely studied circle map is the Arnold sine-circle map, whose dynamics is fairly well known [9]. Previous studies of coupled sine-circle maps have shown spatiotemporal patterns similar to those observed in a damped sine-Gordon partial differential equation with external periodic forcing [10]. Different qualitative classes of dynamical behavior were found: a homogeneous state of periodic behavior, a ''fully developed turbulence'' caused by continuous kink-antikink collisions generating sustained chaotic behavior; and between these extremes a ''soliton turbulence'' regime, characterized by persistence of isolated kink structures in the lattice, the aperiodic behavior being sustained through kink-antikink collisions and pair creation annihilation [10]. Globally coupled circle maps were extensively studied by Kaneko [11], who reported the existence of transitions between coherent, ordered, partially ordered and turbulent phases.

In this paper we are to investigate the frequency synchronization properties of a lattice of coupled sine-circle maps. A recent study has investigated the phase synchronization properties of such systems, by means of an analytic stability method [12]. To each oscillator, or lattice site, we assign a different natural frequency, characterizing a situation that would be expected in the analysis of a turbulence scenario in fluids with isotropic dynamics exhibiting collective quasiperiodic oscillations, in such a way that the frequency is space dependent. Another physical setting in which a distribution of frequencies is possible is the (normal) dispersion of the oscillator frequencies around some mean value. One should determine which conditions must hold in order to ensure or not synchronization even in the presence of frequency nonuniformity. The oscillator frequency after coupling is the map winding number, which is the average phase angle swept in one revolution over the circle.

We suggest a form of lattice coupling, which is described by two parameters: strength and range. The latter is a positive real number that provides a way to pass *continuously* from local nearest-neighbor coupling to global mean-field coupling. The synchronization properties of the coupled map lattice are investigated according to the coupling parameter values. The distribution (mean and deviation) of the perturbed winding numbers, the order parameter magnitude time series and its Fourier spectrum, and the average relative plateau length are used to distinguish between synchronized and nonsynchronized states.

The dependence of the lattice synchronization properties on the coupling range leads to a transition between synchronized and nonsynchronized states as we go from global to local coupling. This transition has also been found in certain nonlinear oscillator chains [13]. The statistical properties of the synchronization plateau sizes are investigated, and an exponential distribution of the plateau lengths is found for a local (nearest-neighbor) form of coupling. Some aspects of this distribution are analyzed, including its thermodynamical limit (infinite lattice size).

This work is organized as follows. In Sec. II we introduce the coupled map lattice to be studied, discussing some of its properties. Section III introduces some diagnostics of synchronization, and discusses the transition between synchronized and non-synchronized states with respect to variations of the coupling parameters. Section IV discusses some statistical aspects of the distribution of synchronization plateau lengths, and Sec. V contains our conclusions.

II. COUPLED SINE-CIRCLE MAP LATTICE

The sine-circle map

$$\theta_{n+1}=f(\theta_n)=\theta_n+\Omega-\left(\frac{K}{2\pi}\right)\sin(2\pi\theta_n) \pmod 1, \quad (1)$$

was proposed by Arnold in 1965 [14] to investigate what would occur with quasiperiodic motion if a nonlinear term (with $K>0$) were added to a rigid rotation map $\theta\mapsto\theta+\Omega$, where $0\leqslant\Omega\leqslant1$ is a ''natural frequency.'' The latter map can model, for example, two periodic oscillators, where θ_n is the phase of one of the oscillators after the other has done one oscillation, and $\Omega\in[0,1)$ is the ratio between the frequencies of the two oscillators [15]. If the frequency Ω is irrational the motion is quasiperiodic, and if Ω is a rational of the form $\tilde{p}/\tilde{q}$ (with $\tilde{p}$, $\tilde{q}$ coprime integers), the motion is periodic with period $\tilde{q}$. These two periodic oscillators may represent, for example, two sinusoidal voltage sources. When they are connected and drive a nonlinear resistor (corresponding to the $K>0$ term), they can self-adjust in a coherent way so that the oscillator basic frequencies become commensurate implying periodic motion, and this frequency locking remains for a certain range of parameters [9]. The parameter space K vs Ω reveals the frequency locking regions as horn-shaped regions, often called Arnold tongues. The periodic motion within a given Arnold tongue can be characterized by a rational value of the winding number, defined as

$$w=\lim_{m\to\infty}\frac{f^m(\theta_0)-\theta_0}{m}, \quad (2)$$

in such a way that we can speak of a $w=\tilde{p}/\tilde{q}$ tongue in parameter space. The winding number is independent of the initial condition θ_0, provided the map $\theta\mapsto f(\theta)$ is invertible.

If $K<1$ the map (1) is always invertible, and the unique motions to be expected in this case are periodic and quasiperiodic, the corresponding regions being densely intertwined in the parameter space. For any nonzero K, the Arnold tongues have finite widths, and no step exists. However, Alstrøm and Ritala [16] have shown that in a lattice of coupled sine-circle maps with random phases a finite nonlinearity $K=K_c$ is needed for a step to appear, and the width of the step scales as $(K-K_c)^2$. If $K>1$, the map iterations may present chaotic behavior, since $f(\theta)$ ceases being invertible and the Arnold tongues overlap [17].

While the temporal behavior of an isolated sine-circle map seems to be fairly well understood, the spatiotemporal evolution of chains of coupled sine-circle maps is still a subject of investigation. Let $\theta_n^{(i)}$ denote the dynamical state in a site i of a one-dimensional lattice of N sites ($i=0,1,\ldots,N-1$) at a time n (with $n=0,1,\ldots$). The time evolution of uncoupled sites is determined by a map $\theta\mapsto f(\theta)$, where $f(\theta)$ is given by Eq. (1). There are many ways to couple these sites in a lattice. One of them is a future additive coupling described by

$$\theta_{n+1}^{(i)}=f(\theta_n^{(i)})+\epsilon[f(\theta_n^{(i+1)})+f(\theta_n^{(i-1)})], \quad (3)$$

where $\epsilon>0$ is the coupling constant. This coupling is local, i.e., it considers only the nearest neighbors of a given site. This system was studied by Crutchfield and Kaneko [10], who reported the existence of soliton turbulence and other phenomena already known for forced sine-Gordon equation. A slightly different, yet much more common, prescription for local coupling is the future Laplacian coupling, which is given by

$$\theta_{n+1}^{(i)}=(1-\epsilon)f(\theta_n^{(i)})+\frac{\epsilon}{2}[f(\theta_n^{(i+1)})+f(\theta_n^{(i-1)})]. \quad (4)$$

On the other hand, it is possible to imagine a global coupling, in which each site couples with a kind of ''mean field'' generated by all other sites, regardless of their relative position on the lattice. This kind of coupling has been used for purposes of chaos control [18], and is given by

$$\theta_{n+1}^{(i)}=(1-\epsilon)f(\theta_n^{(i)})+\frac{\epsilon}{N-1}\sum_{r=1,r\neq i}^{N}f(\theta_n^{(r)}). \quad (5)$$

Between these extremes, it is possible to devise a kind of coupling that decreases with the lattice separation r as a power law $r^{-\alpha}$, where α is a positive real number [13],

$$\theta_{n+1}^{(i)}=(1-\epsilon)f(\theta_n^{(i)})+\frac{\epsilon}{\eta(\alpha)}\sum_{r=1}^{N'}\frac{1}{r^{\alpha}}[f(\theta_n^{(i+r)})+f(\theta_n^{(i-r)})], \quad (6)$$

where η is a normalization factor that interpolates the abovementioned cases

$$\eta(\alpha)=2\sum_{r=1}^{N'} r^{-\alpha}, \tag{7}$$

and $N'=(N-1)/2$. The coupling term is actually a weighted average of discretized spatial second derivatives, the normalization factor being the sum of the corresponding statistical weights. It is straightforward to prove that in the limits $\alpha\to 0$ and $\alpha\to\infty$ Eq. (6) reduces to the global mean-field Eq. (5), and the nearest-neighbor coupling Eq. (4), respectively. So, α may be thought of as a coupling range parameter, which, with the strength ϵ, determines the type of coupling adopted. Further properties of the coupling term for arbitrary α may be found in Ref. [19].

III. TRANSITION BETWEEN SYNCHRONIZED AND NONSYNCHRONIZED STATES

Let us consider now what happens with the perturbed frequencies (winding numbers) of coupled sine-circle maps when their natural frequencies $\Omega^{(i)}$ are randomly distributed over a specified range. In the absence of coupling ($\epsilon=0$) and for zero nonlinearity ($K=0$), each circle map is simply a rigid rotation $\theta^{(i)}\to\theta^{(i)}+\Omega^{(i)}$, and the winding number for each map is simply $\Omega^{(i)}$. Nonlinearity adds entrainment to the system, since for $K\neq 0$ we may find infinitely many Ω values that are locked to a common state with a rational winding number. Coupling between neighboring sites may drive a given map to leave its region in parameter space, for example, it may be driven away from a given Arnold tongue and set down in another tongue, a quasiperiodic, or even a chaotic region. Since for $K>1$ there may be chaotic regions in the parameter space, and since the winding number is ill defined in such cases, we will limit ourselves to values of K far below these regions, and take $K=0.25$ from now on.

The net result of coupling is that, for some maps that have particularly close natural frequencies, the winding numbers

$$w^{(i)}=\lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}|\theta_{n+1}^{(i)}-\theta_n^{(i)}| \tag{8}$$

may be equal, up to a specified tolerance. In this sense we say that these maps have synchronized in frequency. Two or more of such maps would form a synchronization plateau.

We will study the effect of the coupling parameters on the synchronization properties of a lattice of $N=2000$ maps with a variable range coupling given by Eq. (6). We use randomly chosen initial conditions, $\theta_0^{(i)}\in[0,1)$, and periodic boundary conditions, $\theta_n^{(i)}=\theta_n^{(i\pm N)}$. Figure 1, where the coupling strength was held constant at $\epsilon=0.9$, shows the perturbed winding number $w^{(i)}$ as a function of lattice site i, at a fixed time, after transients have died out. Two extreme situations are shown: for $\alpha=0$ (global mean-field coupling) there is complete synchronization at $w=0.5$ [Fig. 1(a)], whereas for $\alpha=3.0$ (which is a quite large value for the coupling range, indicating virtually nearest-neighbor coupling) we see that only a small fraction of the lattice sites has synchronized at $w\approx 0.5$ [Fig. 1(b)].

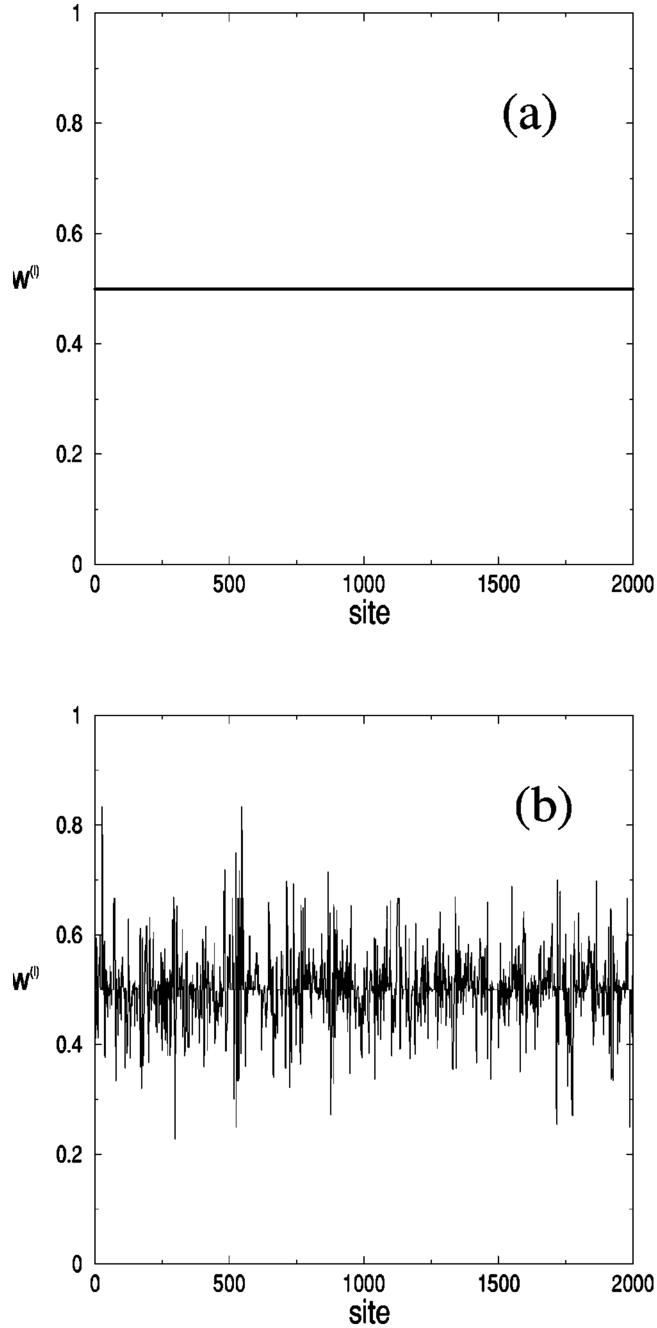

FIG. 1. Winding number profile for a lattice of $N=2000$ coupled sine-circle maps with $K=0.25$, and coupling strength $\epsilon=0.9$. The coupling range is (a) $\alpha=0$ and (b) $\alpha=3$.

A useful diagnostic of synchronization turns out to be the complex order parameter introduced by Kuramoto [4], and here adapted for coupled map lattices. For time n it is defined as [19]

$$z_n=R_n e^{2\pi i\phi_n}\equiv\frac{1}{N}\sum_{j=0}^{N-1}\exp(2\pi i\,\theta_n^{(j)}). \tag{9}$$

Consider two limiting cases: first all sites have the same value of $\theta_n^{(j)}=\xi$ ($j=0,1,2,\ldots,N-1$) leading naturally to a totally synchronized state. One has the order parameter magnitude $R_n=1$ for all times, and a constant phase $\phi_n=\xi$ as well. On the other hand, imagine a pattern in which the site amplitudes $\theta_n^{(j)}$ are so spatially uncorrelated that they may be considered as random variables. So, we may regard the order parameter as the space-averaged factor

$$z_n = \langle e^{2\pi i \theta_n^{(j)}} \rangle_{space} = \langle \cos 2\pi \theta_n^{(j)} \rangle_j + i \langle \sin 2\pi \theta_n^{(j)} \rangle_j = 0. \tag{10}$$

In general, however, it is not necessary that all phases are equal to ensure frequency synchronization. Hence, we expect that for synchronized states the order parameter magnitude R_n has a constant value near unity (for infinite lattices) or fluctuates around a fixed value (for finite chains). Likewise, generic nonsynchronized states are not completely uncorrelated in space, thus they typically exhibit a nonchaotic, yet very irregular oscillation about a *low* value of the order parameter magnitude.

This turns out to be just the case for the two examples previously shown. Figure 2(a) shows the time series for the order parameter magnitude R_n for the same case as in Fig. 1(a). We see regular oscillation of small amplitude around a value near unity, which indeed characterizes a synchronized state. Figure 2(b) shows the corresponding power spectrum, indicating periodic behavior with a sub- and superharmonic structure. Conversely, Fig. 3(a) shows an irregular variation of the order parameter magnitude for the nonsynchronized pattern depicted in Fig. 1(b). The nonchaotic character of these oscillations is best revealed in its power spectrum shown in Fig. 3(b), in which there are many frequency peaks but no broadband noiselike background.

We have found a transition between a completely synchronized and a completely nonsynchronized phase as the coupling range is varied from global to local. Another way to distinguish between these behavior classes is to compute the winding number dispersion around their average $\overline{w} \approx 0.5$

$$\delta w = \left[\frac{1}{N-1} \sum_{i=0}^{N-1} (w^{(i)} - \overline{w})^2 \right]^{1/2}. \tag{11}$$

In Fig. 4(a) we consider a lattice with 2000 sites, fixing the coupling strength at $\epsilon = 0.9$, and varying the coupling range. For low α values the winding number dispersion is near zero, indicating a quite good synchronization behavior. For α slightly higher than 1.0, however, the dispersion jumps to a higher value, and stays there as α increases, indicating that a nonsynchronized state was born.

For another diagnostic, we have defined a synchronization degree by computing the relative mean plateau size for a given winding number profile [13]. Let N_i be the length of the ith synchronization plateau, and let N_p be the total number of plateaus, from which a mean plateau size is $\overline{N}(\alpha) = (1/N_p)\sum_{i=1}^{N_p} N_i$. The synchronization degree will be the ratio between this mean plateau length and the total lattice size, or $p(\alpha) = \overline{N}(\alpha)/N$. For a totally synchronized state, as in Fig. 1(a), we have just one plateau and $\overline{N} = N$, so that $p = 1$. Otherwise, for a completely nonsynchronized state there are almost as many plateaus as sites, so $N_p \approx N$, or $\overline{N} \approx 1$, giving $p \approx 1/N \to 0$ if $N \to \infty$.

In Fig. 4(b) we plot the synchronization degree vs α for the same parameters we have used when computing the winding number dispersion. We see that the transition between a synchronized and a nonsynchronized state occurs for the same critical α as indicated by the jump of the winding number dispersion. A transition like this was already reported for continuous-time oscillator chains [13] and a lattice of coupled kicked oscillators described by circle maps [19]. In order to see why this transition occurs, we should analyze the reasons by which synchronization appears in the lattice due to coupling.

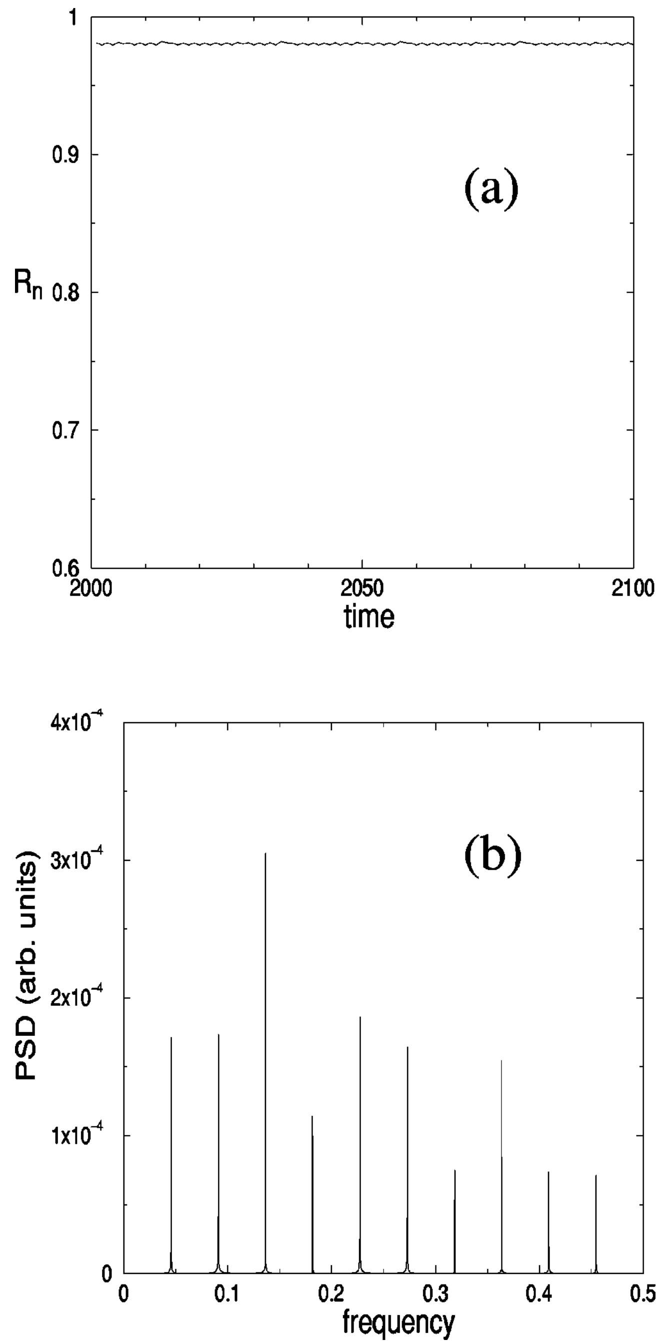

FIG. 2. Order parameter magnitude for $N = 2000$ coupled sine-circle maps with $K = 0.25$, $\epsilon = 0.9$, and $\alpha = 0$. (a) Time series, (b) power spectral density (PSD, arbitrary units).

The distribution of the synchronization plateaus is basically determined by the outcome of a competition between the frozen random disorder represented by the distribution of the natural frequencies $\Omega^{(i)}$ over the lattice, and the diffusion effect caused by coupling. When coupling overcomes the

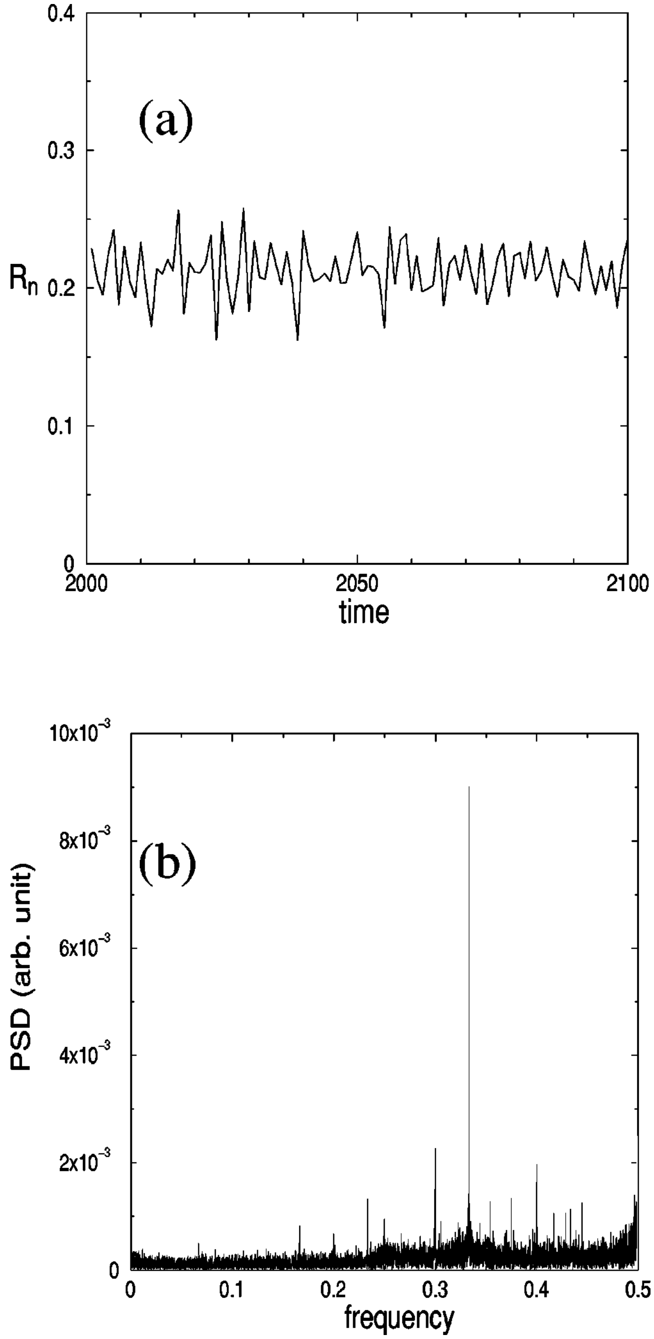

FIG. 3. Order parameter magnitude for $N=2000$ coupled sine-circle maps with $K=0.25,\epsilon=0.9$, and $\alpha=3$. (a) Time series, (b) power spectral density (PSD, arbitrary units).

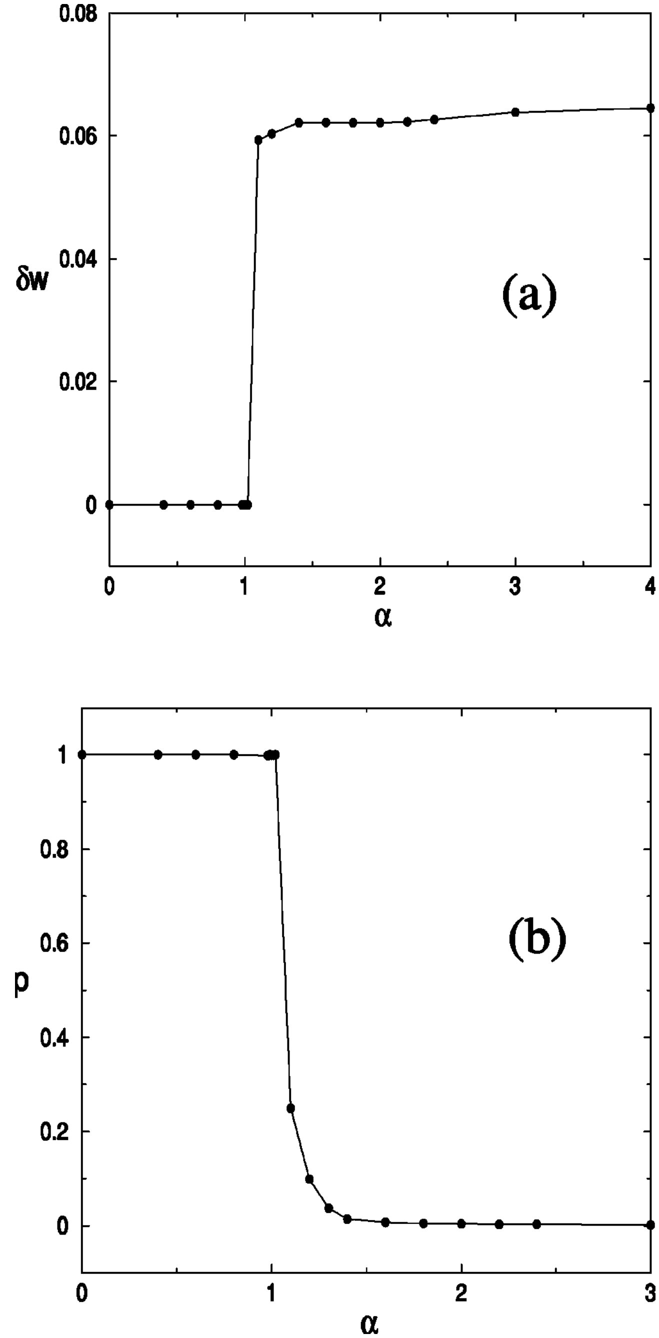

FIG. 4. (a) Winding number dispersion vs coupling range for a coupled lattice of $N=2000$ sine-circle maps with $K=0.25$ and $\epsilon=0.9$; (b) synchronization degree vs coupling range for the same lattice.

frozen disorder in the frequencies, synchronization may occur. For low α each site couples with many other sites, the relative intensity decreasing very slowly with the distance between sites. In this case, the coupling effect is comparatively strong, and in fact it easily overcomes the random disorder making distant sites adjust their perturbed frequencies $w^{(i)}$ to mutually synchronize. On the other hand, a local (large α) coupling connects only nearest neighbors, so its effect is too weak to make distant sites synchronize and form a plateau. The nontrivial aspect of this transition is that it does not look like a smooth process, rather it is a kind of phase transition, with a critical α that depends on the coupling strength and the lattice size.

In order to study this dependence, we have plotted in Fig. 5(a) the winding number dispersion, and in Fig. 5(b) the degree of synchronization as a function of ϵ for a global mean-field lattice ($\alpha=0$). In spite of having a global nature, for small coupling strength its effect is not strong enough to cause synchronization, which is consistent with the limiting case $\epsilon=0$ (uncoupled maps). The transition to a completely synchronized case occurs in an abrupt way.

It must be stressed, however, that p turns out to be a nonrobust statistical quantity, in the sense that it is quite

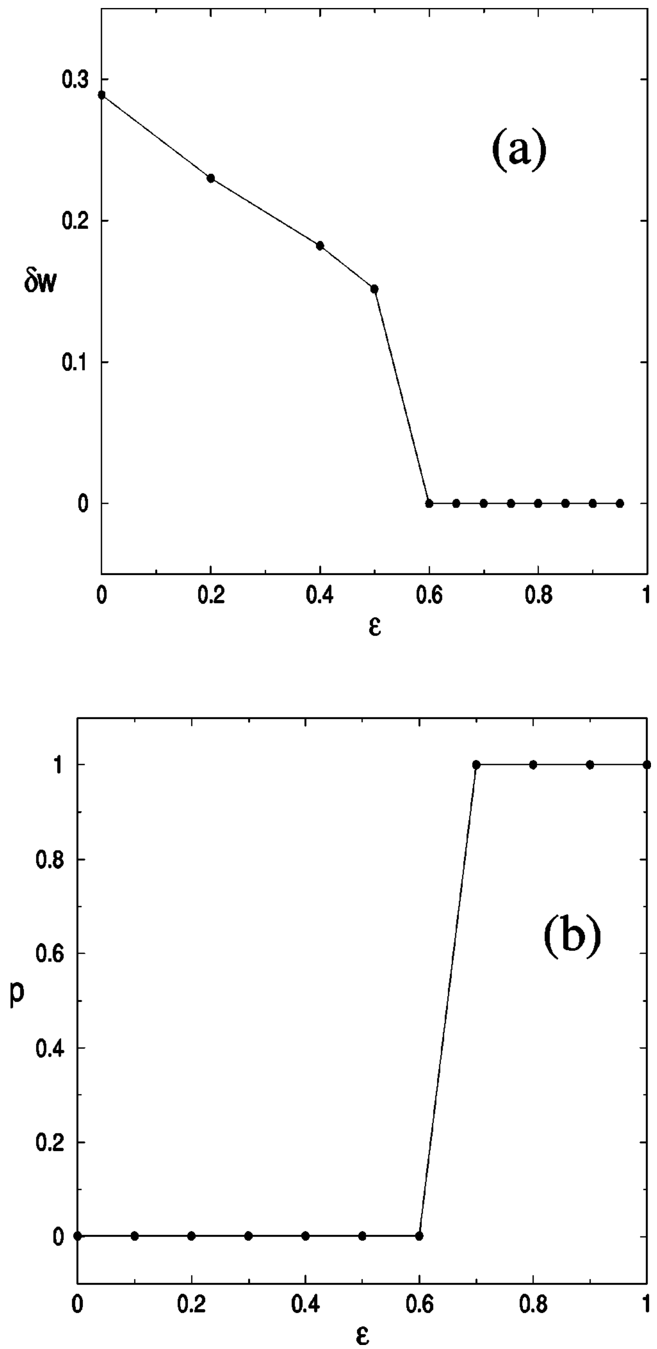

FIG. 5. (a) Winding number dispersion vs coupling strength for a globally coupled lattice ($\alpha = 0$) of $N = 2000$ sine-circle maps with $K = 0.25$; (b) synchronization degree p vs coupling strength for the same lattice.

unstable and may lead to dubious results in certain conditions. For example, if there is only one site that refuses to synchronize in the middle of an otherwise large plateau, it is actually counted twice, regardless of how much that rebel site is deviated from their neighbors. This reduces significantly the synchronization degree, since the number of plateaus may be high without a significant deviation from a totally synchronized state. However, as it occurs with other types of diagnostics, p should be safely used in association with other measures, like the order parameter and the winding number dispersion.

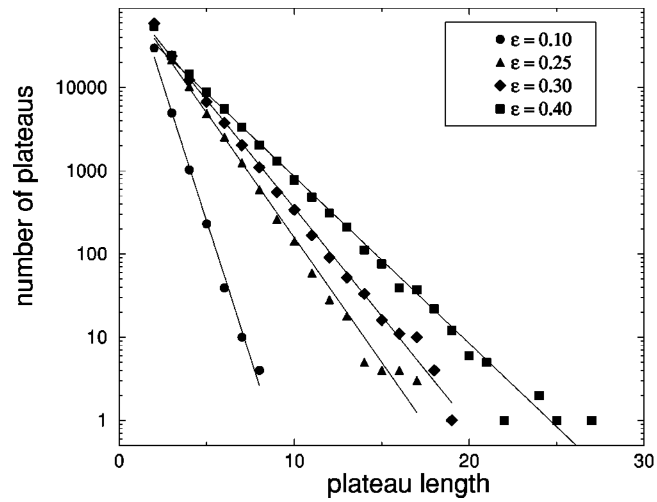

FIG. 6. Number of synchronization plateaus with a given length for a locally (nearest-neighbor) coupled lattice of $N = 800\,000$ sine-circle maps with $K = 0.25$ and different values of the coupling strength. The solid lines represent least-squares fits to the numerical data.

IV. DISTRIBUTION OF PLATEAU LENGTHS

In Sec. III we have dealt with the problem of how to characterize a given synchronization regime, and the transition between different regions, by using some relevant quantities. However, up to now we did not consider in detail the distribution of plateaus over the lattice, in particular its dependence with the plateau length.

As we have seen, global couplings tend to favor large plateaus, so that we will use hereafter a nearest-neighbor coupling (corresponding to $\alpha \to \infty$), in order to get a reasonable number of small plateaus. In the same way, statistics is better if we have a large total number of sites N. However, very large lattices are difficult to deal with in a scheme that

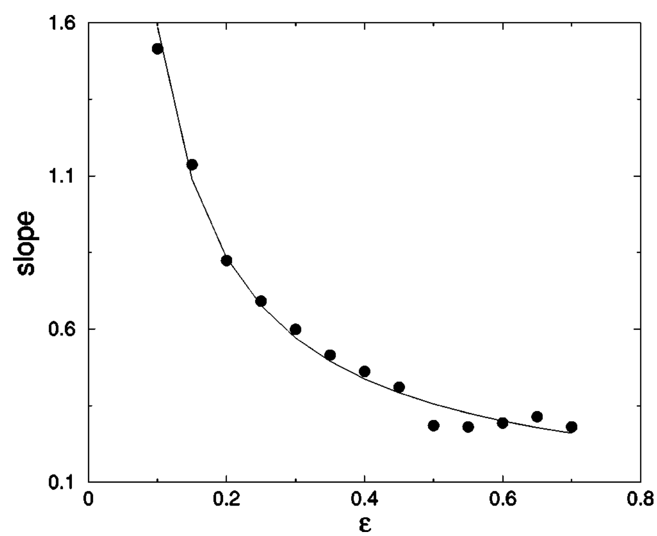

FIG. 7. Slope of the synchronization plateau distribution vs coupling strength for a lattice of $N = 800\,000$ coupled sine-circle maps with $K = 0.25$. The solid curve is a power-law fitting of numerical data.

uses variable range, so that local couplings furnish good results with less computer time [20]. In this way we will be able to work with lattices up to 800 000 sites.

Figure 6 shows frequency histograms (semilog plots) of the number of synchronization plateaus n vs their lengths N_i, for some values of the coupling strength ϵ, and using a nearest-neighbor coupling. These distributions are well fitted by straight lines, indicating an exponential-type dependence

$$n(N_i,\epsilon)=ke^{-m(\epsilon)N_i}, \tag{12}$$

with k and m positive and real parameters. For small coupling ($\epsilon=0.1$), it is different to find very large clusters. Indeed, there are less than ten plateaus with length equal to 7 (in a lattice of 800 000 sites) and no larger plateaus. The major part of them has very small lengths. The distribution slope is $m=1.514\pm0.053$.

For a larger coupling strength ($\epsilon=0.25$) we may see the presence of larger plateaus, with lengths up to almost 20 sites, and a smaller slope (0.690 ± 0.020). Large plateaus are more rare, thus contributing to a poor statistics in this region, which is clearly seen in the point scattering for larger N_i. The best linear fit is thus obtained for smaller plateaus, and in this region the slope becomes smaller as ϵ increases.

The characteristic parameter of the plateau length distribution turns out to be the histogram slope m. The variation of m with the coupling strength ϵ is depicted in Fig. 7, where we observe a monotonic decrease of the slope as ϵ increases from 0.1 (very weak coupling) to 0.7 (moderately strong coupling), appearing to saturate on ≈0.28 for the latter case. Whether or not this behavior would persist for larger ϵ is not clear though, since statistics tends to be worse due to the formation of a progressively smaller number of large plateaus (the coupling is intense). We have observed a power-law dependence for this case, in the form $m(\epsilon)\sim\epsilon^{-\beta}$, with $\beta=0.929\pm0.047$.

Even though we are dealing with large lattices in order to ensure good statistics, it turns out to be important to investigate the thermodynamical limit ($N\rightarrow\infty$) of the lattice. In Fig. 8 we have plotted the distribution slope vs lattice size N for two values of the coupling strength ϵ. For both values of ϵ we note that the distribution slope appears to converge to a value of ≈1.51 (for $\epsilon=0.1$) and ≈0.59 (for $\epsilon=0.3$). Since these values do not change appreciably from *circa* 200 000 sites to 800 000 sites, we expect that the previous results are valid, with a good approximation, in the thermodynamical limit. This strongly indicates that the plateau length distribution is of an exponential type in the thermodynamical limit of the coupled map lattice.

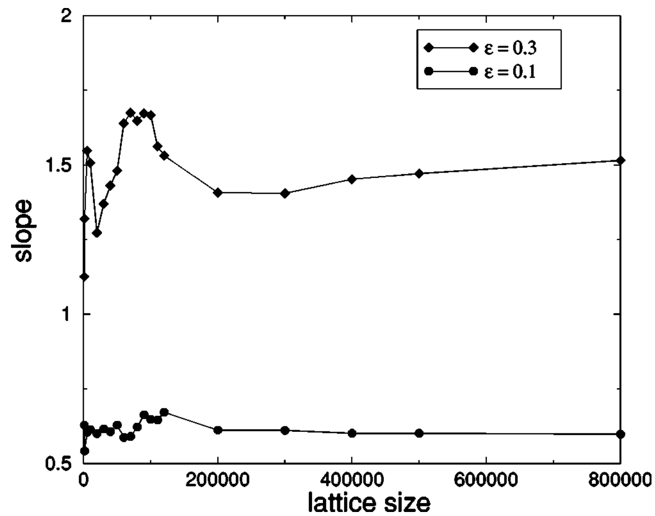

FIG. 8. Slope of the synchronization plateau distribution vs lattice size for two values of the coupling strength, in a lattice of $N=800\,000$ coupled sine-circle maps with $K=0.25$.

V. CONCLUSIONS

We have studied in this paper the formation of synchronization plateaus in lattices of coupled sine-circle maps. These maps were chosen because of their importance in the modeling of spatially extended coupled oscillator systems, like arrays of Josephson junctions or chains of coupled driven pendula. Each map was given a randomly distributed natural frequency, and the lattice coupling was described by two parameters: its strength and range. The latter parameter is capable of conveying information about the dependence of the coupling properties with the distance between adjacent sites (in our case, this influence decays in a power-law fashion).

Some diagnostics of synchronization were introduced. The dispersion of the winding numbers (perturbed frequencies) around their average, the complex order parameter, and the relative mean plateau size were used to distinguish between synchronized and nonsynchronized states. We have observed a transition between a completely synchronized and a totally unsynchronized lattice as the coupling range is varied from global (mean field) to local (nearest neighbor).

The principle underlying this transitional behavior is the competition between two opposite forces: the frozen random disorder caused by the distribution of the natural frequencies over a given domain and the diffusion effect caused by coupling. When the latter overcomes the former cause, we have a tendency of large-scale synchronization. Otherwise, the lattice remains unsynchronized even in presence of coupling.

The distribution of the synchronization plateaus according to their sizes was found to be of a decreasing exponential nature, when the coupling is strictly local (nearest neighbor), i.e., there are far more small sized plateaus than larger ones. The characteristic parameter of this exponential distribution (its slope) was found to decrease with the coupling strength, and it relaxes to a constant value for very large lattices. We have considered sufficiently large lattices in our simulations, so that we claim that these distribution properties are valid in the thermodynamical limit.

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