Plasma Phys. Control. Fusion **41** (1999) 567–573. Printed in the UK PII: S0741-3335(99)98648-7

On axisymmetric double adiabatic MHD equilibria with plasma flow

Roberto A Clemente† and Ricardo L Viana‡

Instituto de Fisica Gleb Wataghin, Universidade Estadual de Campinas, 13083-970 Campinas, SP, Brazil

Departamento de Física, Universidade Federal do Paraná, 81531-990, Curitiba, PR, Brazil

Received 23 October 1998, in final form 21 January 1999

Abstract. The stationary equilibrium of an axisymmetric plasma characterized by toroidal and poloidal flows is considered within the framework of ideal double adiabatic magnetohydrodynamic equations. The problem is reduced to a nonlinear partial differential equation for the poloidal magnetic flux function, containing six surface functions, plus a nonlinear algebraic Bernoulli equation defining the plasma density. Ellipticity conditions and bifurcations of its solutions are discussed in the limit of small beta, appropriated for tokamak-like equilibria. Possible connections with the L–H transition are suggested.

1. Introduction

At the end of the 1970s, various tokamak experiments, in which neutral beam heating was applied, indicated the existence of an important velocity field in the plasma and also some kind of pressure anisotropy (Bell 1979, Suckewer *et al* 1979). Almost all regimes of tokamak operation present some sort of plasma flow, sometimes with very large Mach numbers. Both toroidal and poloidal plasma velocities have been observed (Brau *et al* 1983, Isler 1983, Taylor *et al* 1991). Plasma flow is of relevance also in other magnetic confinement schemes, e.g. field-reversed configurations (Carraro *et al* 1998), where azimuthal rotation is responsible for an $n = 2$ instability that may destroy plasma confinement (Linford *et al* 1979).

A basic step to approach studies on the effects of plasma flow and anisotropy on macroscopic transport and stability related problems should be to find a realistic axisymmetric equilibrium model. The problem of axisymmetric magnetohydrodynamic (MHD) equilibria with isotropic plasma pressure and general flow has been studied by several authors (Zehrfeld and Green 1970, Morozov and Solov'ev 1980, Hameiri 1983, Maschke and Perrin 1984, Kerner and Tokuda 1987, Zelazny *et al* 1993, Tasso and Throumoulopoulos 1998). In general, the problem is reduced to two coupled equations, one nonlinear partial differential equation for the poloidal magnetic flux function, containing hypotheses on five surface quantities and a nonlinear algebraic Bernoulli equation defining plasma density.

This makes the problem quite intractable and the only known approximate analytical solution is due to Maschke and Perrin (1984), who have considered the limit of small plasma beta and small ratio of poloidal to toroidal magnetic field. Żelazny and collaborators (Żelazny *et al* 1993) have obtained numerical solutions of the equations by using the inverse method and Fourier decomposition. In agreement with the results of Hameiri (1983), they quantitatively showed the existence of different branches for the plasma density and the special behaviour of the nonlinear partial differential equation, which alternates between elliptic and hyperbolic, when the Alfvén–Mach number of the poloidal flow with respect to the poloidal magnetic field increases. The existence of hyperbolic regimes implies that some kind of shock wave should be generated in the plasma and no closed equilibrium would be possible.

On the other hand, if the plasma is subjected to a strong magnetic field, ions have a small Larmor radius and a rapid gyro-motion. This gives rise to pressure anisotropy and it should be taken into account if a general MHD equilibrium equation is to be written down.

In this paper we present more general MHD equilibrium equations, which consider both poloidal and toroidal flows as well as pressure anisotropy. To pursue this task we will apply double adiabatic equations of state, in which different plasma pressures—along and orthogonally to the magnetic field—are allowed. In analogy with simple adiabatic models, a nonlinear partial differential equation for the poloidal magnetic flux is obtained in terms of six surface functions (one more than in the adiabatic case). Such an equation is coupled to a nonlinear algebraic Bernoulli equation defining the plasma density.

Since double adiabatic equations of state loose their meaning when the magnetic field vanishes, toroidal equilibria like FRC (field-reversed configurations, in which the magnetic field vanishes at the O-point) with poloidal plasma flow cannot be described by the present model. However, this does not invalidate previous results on the existence of static and rotating FRC with pressure anisotropy, since the state equations do not enter in such models (Clemente 1993, 1994). As a bypass result that has not been pointed out in previous studies, we show that in the case of FRC with simple adiabatic flow and isotropic pressure, both toroidal and poloidal flow cannot coexist simultaneously.

Our Bernoulli equation presents some advantage over the corresponding Bernoulli equation for simple adiabatic flows, since it can be reduced to two auxiliary equations defining different branches for the density. Moreover, at least one of the bifurcation points for the density can be obtained analytically. It is shown that such a bifurcation point falls inside the second region of ellipticity of the nonlinear partial differential equation for the poloidal magnetic flux. The critical points for the elliptic regimes are obtained analytically in the case of small thermal to magnetic energy ratio and small poloidal to magnetic field ratio. The bifurcation behaviour of the density should be connected with the L–H transition in tokamaks, which is still not well understood.

This paper is organized as follows: in section 2 the equilibrium model in cylindrical coordinates is presented. The model equations and the magnetic field and velocity representations are worked out to give the momentum balance partial differential equation and the generalized Bernoulli equation. The final section is devoted to a discussion on ellipticity and density bifurcation conditions, as well as to our conclusions.

2. Stationary double adiabatic model

We will consider an ideal plasma of electrons and singly charged ions in stationary equilibrium. All partial time derivatives will be assumed to vanish, whereas plasma pressure anisotropy and a non-zero plasma velocity will be allowed. In the following, cylindrical coordinates (r, z, ϕ) and axisymmetry will be assumed throughout. Standard units will be adopted and the Chew–Goldberger–Low form for the pressure tensor will be considered (Chew *et al* 1956)

$$\mathbb{P} = p_\perp \mathbb{I} + \sigma_- \boldsymbol{B}\boldsymbol{B} \tag{1}$$

where $\mathbb{I}$ is the identity tensor, $p_\parallel$ and $p_\perp$ are parallel and perpendicular plasma pressures, respectively, $\boldsymbol{B}$ is the magnetic field, $\sigma_- = (p_\parallel - p_\perp)/B^2$ is a measure of pressure anisotropy and $B = |\boldsymbol{B}|$.

The corresponding ideal MHD equations for stationary equilibria are

$$\nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{2}$$

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla \cdot \mathbb{P} = \boldsymbol{j} \times \boldsymbol{B} \tag{3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{4}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{j} \tag{5}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{6}$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{7}$$

If the plasma is subjected to a strong magnetic field, it may present distinct perpendicular and parallel pressures that obey different evolution equations, the so-called double adiabatic equations (Stacey 1981)

$$\boldsymbol{v} \cdot \nabla \ln p_{\parallel} - \boldsymbol{v} \cdot \nabla \ln \rho + \frac{2\boldsymbol{B} \cdot [(\boldsymbol{B} \cdot \nabla)\boldsymbol{v}]}{B^2} = 0 \tag{8}$$

$$\boldsymbol{v} \cdot \nabla \ln p_{\perp} - 2\boldsymbol{v} \cdot \nabla \ln \rho - \frac{\boldsymbol{B} \cdot [(\boldsymbol{B} \cdot \nabla)\boldsymbol{v}]}{B^2} = 0 \tag{9}$$

where

$$\rho = n(m_{\mathrm{e}} + m_{\mathrm{i}}) \tag{10}$$

is the plasma mass density, n being the particle number density and m_{e}, m_{i} the electronic and ionic masses, respectively. $\boldsymbol{v}$, $\boldsymbol{E}$, and $\boldsymbol{j}$ are the plasma velocity, electric field, and plasma current density, respectively.

In axisymmetric equilibria the magnetic field can be represented in terms of two scalar functions $\Psi(r, z)$ and $I(r, z)$ (poloidal magnetic flux and current functions, respectively)

$$\boldsymbol{B} = \nabla\Psi \times \frac{\hat{\boldsymbol{e}}_\phi}{r} + I\frac{\hat{\boldsymbol{e}}_\phi}{r}. \tag{11}$$

According to equation (2), introducing two additional scalar functions $\Gamma(r, z)$ and $F(r, z)$, a similar representation can be used for $\rho\boldsymbol{v}$,

$$\rho\boldsymbol{v} = \nabla F \times \frac{\hat{\boldsymbol{e}}_\phi}{r} + \rho\Gamma\frac{\hat{\boldsymbol{e}}_\phi}{r}. \tag{12}$$

Taking the cross product between (12) and (11), we obtain

$$\boldsymbol{v} \times \boldsymbol{B} = \frac{\nabla\Psi \times \nabla F}{r^2} - \frac{I\nabla F}{\rho r^2} + \frac{\Gamma\nabla\Psi}{r^2}. \tag{13}$$

Since, from Faraday's law (6), the azimuthal component of $\boldsymbol{E}$ must vanish, the Jacobian of the transformation $(r, z) \to (F, \Psi)$ must be identically zero, i.e. $F = F(\Psi)$ is a surface function. This means that magnetic field lines and flow lines lie on the same nested toroidal surfaces and for any surface quantity Q it follows that $\boldsymbol{B} \cdot \nabla Q = \boldsymbol{v} \cdot \nabla Q = 0$. Hence $\nabla F = F'\nabla\Psi$, where from here on the prime will denote differentiation with respect to Ψ.

Taking the curl of (13) it is possible to show that

$$\Omega = \frac{\Gamma}{r^2} - \frac{IF'}{\rho r^2} \tag{14}$$

is also a surface quantity, physically being a toroidal angular velocity ($\Omega = v_\phi/r$). Combining (14) and (15) we have the following expression for the plasma velocity,

$$\boldsymbol{v}(r, z) = \frac{F'}{\rho}\boldsymbol{B}(r, z) + \Omega r\hat{\boldsymbol{e}}_\phi. \tag{15}$$

The momentum balance equation (3) can be rearranged in the following way

$$\nabla\left(p_\perp + \frac{B^2}{2}\right) + \nabla\cdot\mathbb{T} = 0 \tag{16}$$

where we have defined the tensor

$$\mathbb{T} = (\sigma_- - 1)\boldsymbol{BB} + \rho\boldsymbol{vv}. \tag{17}$$

Since (16) has vanishing azimuthal component, a new surface quantity results

$$\Lambda(\Psi) = (\sigma_- - 1)I + F'\Gamma. \tag{18}$$

From the starting set of equations it is possible to obtain an energy theorem for anisotropic stationary equilibria

$$\nabla\cdot\left[\left(\frac{p_\parallel}{2\rho} + \frac{2p_\perp}{\rho} + \frac{v^2}{2}\right)\rho\boldsymbol{v} + \sigma_-\boldsymbol{B}(\boldsymbol{B}\cdot\boldsymbol{v}) + \boldsymbol{E}\times\boldsymbol{B}\right] = 0 \tag{19}$$

which can be written as

$$\boldsymbol{B}\cdot\nabla\left[\frac{3p_\parallel}{2\rho} + \frac{p_\perp}{\rho} + \frac{v^2}{2} - \frac{\Omega I}{F'}(1-\sigma_-)\right] = 0 \tag{20}$$

from which we recognize the existence of another surface quantity:

$$\Theta = \frac{3p_\parallel}{2\rho} + \frac{p_\perp}{\rho} + \frac{F'^2B^2}{2\rho^2} - \frac{\Omega^2 r^2}{2}. \tag{21}$$

$\Theta(\Psi)$ represents a generalized Bernoulli law. It may be seen that for $\sigma_- = 0$ it reduces to the previously obtained Bernoulli law for isotropic pressure and adiabatic flow with specific heat ratio $\gamma = 5/3$ (Maschke and Perrin 1984).

Using (1), (8), and (9) we can also obtain

$$\nabla\cdot\left(\frac{p_\parallel p_\perp^2}{\rho^4}\boldsymbol{v}\right) = 0 \tag{22}$$

$$\nabla\cdot\left(\frac{p_\perp}{B}\boldsymbol{v}\right) = 0 \tag{23}$$

$$\nabla\cdot\left(\frac{p_\parallel B^2}{\rho^2}\boldsymbol{v}\right) = 0 \tag{24}$$

from which the existence of two other surface quantities

$$g(\Psi) = \frac{p_\perp}{\rho B} \qquad f(\Psi) = \frac{p_\parallel B^2}{\rho^3} \tag{25}$$

may be recognized.

At this point, by considering the component parallel to $\nabla\Psi$ of the momentum balance equation (3), and after some straightforward algebra, we are able to obtain a partial differential equation for the poloidal magnetic flux function as

$$\begin{aligned}\left(1 - \frac{F'^2}{\rho} - \sigma_-\right)\Delta^*\Psi = &-r^4\rho\Omega\Omega' - I\Lambda' - r^2(F'\Omega)'I - r^2\rho\Theta' \\ &+ I^2\frac{F'F''}{\rho} + \nabla\Psi\cdot\left[F'\nabla\left(\frac{F'}{\rho}\right) + \nabla\sigma_-\right] + r^2\left(p_\perp\frac{g'}{g} + \frac{p_\parallel}{2}\frac{f'}{f}\right)\end{aligned} \tag{26}$$

where

$$\Delta^*\Psi = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) + \frac{\partial^2\Psi}{\partial z^2} \tag{27}$$

is the well known Grad–Shafranov operator.

Equation (26) together with (21) form a coupled set of equations whose solutions depend on six hypotheses on the functional dependence with Ψ of the surface quantities Ω, F', f, g, Λ, and Θ, the plasma mass density being defined by (21).

3. Discussion

The equations obtained in the previous section allow us to treat the general problem of stationary axisymmetric anisotropic MHD equilibria. All previous axisymmetric MHD equilibrium models can be recovered from this set of equations. The isotropic pressure and adiabatic flow axisymmetric equilibria are described when σ_- vanishes and $\gamma = 5/3$ (Morozov and Solovev 1980, Hameiri 1983, Maschke and Perrin 1984, Kerner and Tokuda 1987). The equations for axisymmetric plasmas with pure toroidal flow and anisotropic pressure are recovered when F' vanishes (Mercier and Cotsaftis 1961, Clemente 1993, 1994). Isotropic pressure equilibria with pure toroidal rotation are described when $F' = \sigma_- = 0$ (Maschke and Perrin 1980, Clemente and Farengo 1984, Viana *et al* 1997).

Some remarks are needed in the case of purely poloidal magnetic fields (FRC for example). In such a case some region of vanishing magnetic field, where $p_\parallel = p_\perp$, will necessarily exist, while double adiabatic equations predict different expressions for the density. Such expressions will also be incompatible with the corresponding density arising from Bernoulli equation (21). Our conclusion is that, when poloidal flow is present, FRC-like equilibria cannot be anisotropic in the Chew, Golberger and Low sense. However, they can be anisotropic if only toroidal rotation is present, as has recently been shown (Clemente 1994).

Another interesting point, which has not been pointed out previously, comes out from considering the expression for I arising from (18), when it must vanish with $\sigma_- = 0$, i.e. isotropic pressure and simple adiabatic flow. It can be seen that $I = 0$ may be fulfilled only when $F' = 0$ or $\Omega = 0$, or both are vanishing simultaneously. This means that in an axisymmetric FRC the coexistence of poloidal and toroidal flow is not admitted within a simple adiabatic ideal MHD model.

Due to the previous considerations, from here on the discussion will be restricted to toroidal equilibria for which both poloidal and toroidal magnetic fields exist and there are no regions of vanishing magnetic field in the plasma.

An important issue formerly discussed by Hameiri (1983), in the case of isotropic plasma pressure and simple adiabatic flow, are the conditions of ellipticity for the nonlinear partial differential equation for the poloidal magnetic flux. Taking into account that σ_- and the density arising from Bernoulli law (21) are both functions of Ψ, $|\nabla\Psi|^2$, and r, the determinant of the symmetric matrix of the coefficients of the second-order derivatives of Ψ in equation (26) can be obtained as

$$\mathrm{Det} = \left(1 - \frac{F'^2}{\rho} - \sigma_-\right)^2 \left[\frac{1 - (F'^2/\rho) - \sigma_- - X}{1 - (F'^2/\rho) - \sigma_- - X(1 - (|\nabla\Psi|^2/r^2B^2)}\right] \quad (28)$$

where

$$X = \frac{(3\beta_\parallel - \beta_\perp - F'^2/\rho)^2}{(3\beta_\parallel - F'^2/\rho)} - 4\beta_\parallel + \beta_\perp \quad (29)$$

with

$$\beta_\parallel = \frac{p_\parallel}{B^2} \qquad \beta_\perp = \frac{p_\perp}{B^2}. \quad (30)$$

Equation (26) will be elliptic if Det > 0, and in this case a boundary value problem is well defined for the equilibrium equations.

Thinking in tokamak (low-beta) applications, we can develop the denominator of Det to the first order in $|\nabla\Psi|^2/r^2B^2$, $\beta_\parallel$, and $\beta_\perp$. It can be seen that Det is positively definite when the following conditions hold:

$$0 < \frac{F'^2}{\rho} < 3\beta_\parallel - \frac{\beta_\perp^2}{1 + 2\beta_\perp}$$

$$3\beta_\parallel - \frac{\beta_\perp^2}{1+2\beta_\perp}\left(1 - \frac{|\nabla\Psi|^2}{r^2B^2}\right) \lessapprox \frac{F'^2}{\rho} < 1 - \beta_\parallel + \beta_\perp$$
$$1 - \beta_\parallel + \beta_\perp < \frac{F'^2}{\rho} \lessapprox \frac{r^2B^2}{|\nabla\Psi|^2}.$$

As in the simple adiabatic case we have three regions of ellipticity for the parameter F'^2/ρ, which can be interpreted as the square of the Alfvén–Mach number. We note that the second region is separated from the first one by a very small gap.

At this point it is interesting to note that in the present case of double adiabatic flow, using the definition of f and g, the Bernoulli equation may be split into two auxiliary equations for the density,

$$\frac{3f\rho^2}{B^2} = \Theta + \frac{\Omega^2r^2}{2} - gB \pm \sqrt{\left(\Theta + \frac{\Omega^2r^2}{2} - gB\right)^2 - 3fF'^2} \tag{31}$$

where it must be taken into account that B also depends on ρ. The plus or minus sign distinguishes different branches for ρ. Their equivalence defines one of the bifurcation points for the density.

Using expression (21) for Θ, the vanishing of the square root in (31) defines a bifurcation point for the density corresponding to $F'^2/\rho = 3\beta_\parallel$. As can be seen from the ellipticity conditions, such a bifurcation point falls just inside the second region of ellipticity for the nonlinear partial differential equation for the poloidal magnetic flux. As the plasma flows may drive anisotropy, its connection with a transition similar to the L–H transition in tokamaks is quite suggestive.

In conclusion, we have presented a new set of equations to describe axisymmetric stationary equilibria with anisotropic pressure, which reduce to the equilibrium equations already known in the appropriate limits. Conditions of ellipticity of the resulting nonlinear partial differential equation were shown, together with possible bifurcations for the density. Further work will be needed in order to obtain solutions which will be useful in describing plasma configurations appearing in the present generation of tokamaks.

Acknowledgments

The authors would like to thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil), Fundação de Amparo à Pesquisa do Estado de São Paulo (Fapesp, Brazil) and National Science Foundation (NSF, USA through a joint research project) for financial support.

References

Bell M G 1979 *Nucl. Fusion* **19** 33
Brau K, Bitter M, Goldston R J, Manos D, McGuire K and Suckewer S 1983 *Nucl. Fusion* **23** 1643
Carraro L, Puiatti M E, Sattin F, Scarin P and Valisa M 1998 *Plasma Phys. Control. Fusion* **40** 1021
Chew G F, Goldberger M L and Low F E 1956 *Proc. R. Soc.* A **236** 112
Clemente R A 1993 *Nucl. Fusion* **33** 963
——1994 *Plasma Phys. Control. Fusion* **36** 707
Clemente R A and Farengo R 1984 *Phys. Fluids* **27** 776
Hameiri E 1983 *Phys. Fluids* **26** 230
Isler R C *et al* 1983 *Nucl. Fusion* **23** 1017
Kerner W and Tokuda S 1987 *Z. Naturf.* a **42** 1154
Linford R K, Armstrong W T, Platts D A and Sherwood E G 1979 *Proc. IAEA Conf. on Plasma Physics and Controlled Nuclear Fusion Research* vol II (Vienna: IAEA) p 447

Maschke E K and Perrin H 1980 *Plasma. Phys.* **22** 579
——1984 *Phys. Lett.* A **102** 106
Mercier C and Cotsaftis M 1961 *Nucl. Fusion* **1** 121
Morozov A I and Solov'ev L S 1980 *Reviews of Plasma Physics* vol 8, ed M A Leontovich (New York: Consultants Bureau) ch 2
Suckewer S, Eubank H P, Goldston R J, Hinnov E and Sauthoff N R 1979 *Phys. Rev. Lett.* **43** 207
Stacey W M Jr 1981 *Fusion Plasma Analysis* (New York: Wiley)
Taylor R *et al* 1991 *Proc. 13th IAEA Conf. on Plasma Physics and Controlled Nuclear Fusion Research (Washington, 1990)* (Vienna: IAEA) paper CN-53/A-6-5
Tasso H and Throumoulopoulos G N 1998 *Phys. Plasmas* **5** 2378
Viana R L, Clemente R A and Lopes S R 1997 *Plasma Phys. Control. Fusion* **39** 197
Zehrfeld H P and Green B J 1970 *Nucl. Fusion* **10** 251
Żelazny Z, Stankiewicz R, Gallkowski A and Potempski S 1993 *Plasma Phys. Control. Fusion* **35** 1215