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Spherically symmetric stationary MHD equilibria with azimuthal rotation

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Abstract. An equation for stationary MHD equilibrium with azimuthal rotation in spherical coordinates is derived. Following a procedure introduced by Maschke and Perrin, by supposing that the plasma temperature is a surface quantity, we describe a two-parameter family of axisymmetric toroidal configurations. The equilibrium equation is analytically solved for given hypotheses in such a way as to investigate the effect of plasma rotation on the magnetic field structure.

1. Introduction

Azimuthal rotation in magnetically confined plasmas has been intensively studied in many problems in plasma physics. In fusion plasmas, it may appear as a result of neutral beam injection in tokamaks (Bell 1979, Suckewer *et al* 1979). In astrophysical plasmas, the problem of rotating magnetic stars also presents such aspects (Plumpton and Ferraro, 1955).

The plasma is described by the ideal MHD equations. The stationary equilibrium state is characterized by the vanishing of all time derivatives; a constant azimuthal velocity is allowed. In accordance with Alfvén's theorem, the magnetic field lines (and the magnetic flux surfaces) rotate rigidly with the plasma. As in magnetostatic equilibria, we use scalar surface quantities in order to describe magnetic surfaces—the poloidal magnetic flux, and the poloidal current function. The MHD equations reduce to an elliptic partial differential equation, which is known as Grad–Shafranov–Schlüter equation in static equilibria (Wesson, 1987).

For stationary axisymmetric toroidal equilibria with azimuthal rotation, a similar equation was obtained by Maschke and Perrin (1980). They assumed that the thermal conductivity is higher along the magnetic field lines than across them, so temperature is taken as a surface quantity too. A corresponding equation has also been derived considering the entropy as a surface quantity.

The equation derived by Maschke and Perrin holds only for cylindrical coordinates. It is more difficult to solve, from an analytical point of view, than the corresponding static Grad–Shafranov–Schlüter equation. Only three analytical solutions of it are found in the literature. One of them is due to Maschke and Perrin themselves, the other two being presented by Missiato and Sudano (1982) and Clemente and Farengo (1984).

However, cylindrical geometry is rather inadequate for dealing with spherically symmetric problems. In fusion plasmas these problems appear in compact tori models, like spheromaks and field-reversed configurations (FRC) (Turner, 1984). An interesting type of such equilibria has been found by Morikawa (1969), dealing with configurations enclosed in a spherical conducting shell. This model has recently been studied from a field line Hamiltonian approach (Viana, 1995). Morikawa and Rebhan (1970) have considered a similar family of solutions surrounded by a vacuum field. These problems are also of interest in astrophysics, and analytical solutions for MHD equilibria in magnetic stars have been studied since the early work of Prendergast (1956).

A version of the Maschke–Perrin equation for spherically symmetric rotating equilibria is thus required for problems of interest in both fusion and astrophysical plasmas. In this paper, we present such an equation, derived by choosing the temperature as a surface quantity.

This paper is organized as follows: in section 2, we present the procedure due to Maschke and Perrin for including the rotation effects. We discuss in section 3 an analytical solution for a spherical plasma contained in a perfectly conducting shell. Section 4 is devoted to a discussion of some properties of the solution, our conclusions being left to the final section.

2. Equilibrium equation with azimuthal rotation

Let us consider an ideal plasma of singly charged ions in stationary equilibrium ($\partial/\partial t \equiv 0, \boldsymbol{v} \neq 0$) in which the plasma velocity has only an azimuthal component. Using spherical coordinates (r, θ, ϕ), when axisymmetry is assumed the magnetic field can be expressed in terms of the usual magnetic stream function $\Psi(r, \theta)$ as

$$\boldsymbol{B}(r,\theta) = \frac{\hat{\phi} \times \nabla\Psi(r,\theta)}{r\sin\theta} - \frac{\mu_0 I(r,\theta)}{r\sin\theta}\hat{\phi} \tag{1}$$

where $I(\Psi)$ is the poloidal current function and μ_0 the vacuum magnetic permeability. Correspondingly, if $\boldsymbol{v} = \Omega(r,\theta) r\sin\theta\hat{\phi}$, from Faraday's law and ideal Ohm's law it follows that $\Omega = \Omega(\Psi)$, i.e. the isorotation law (Ferraro, 1937).

Then, the momentum balance equation for the plasma reduces to

$$\nabla p - \rho\Omega^2 r\sin\theta(\sin\theta\hat{r} - r\cos\theta\hat{\theta}) = -\left[\frac{\Delta^*\Psi}{\mu_0 r^2\sin^2\theta} + \frac{\mu_0(\mathrm{d}I^2/\mathrm{d}\Psi)}{2r^2\sin^2\theta}\right]\nabla\Psi \tag{2}$$

where ρ is the plasma mass density, p the kinetic pressure and

$$\Delta^* \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

is the Grad–Shafranov operator in spherical coordinates.

In order to further reduce the equation, some additional assumptions are needed. If p obeys an ideal gas law of the type

$$p = \frac{\rho k}{(m_{\mathrm{e}} + m_{\mathrm{i}})}T$$

where $T = T_{\mathrm{e}} + T_{\mathrm{i}}$ is the plasma temperature (sum of electronic and ionic temperatures), k is Boltzmann's constant, and $T = T(\Psi)$ is assumed (this means that the thermal conductivity of the plasma is much larger along magnetic field lines than across them). It is possible to show, by taking the cross product of equation (2) with $\nabla\Psi$, that another surface function $g(\Psi)$ exists, given by

$$g(\Psi) \equiv p(r,\theta)\ \exp\left[-\frac{(m_{\mathrm{e}} + m_{\mathrm{i}})\Omega^2 r^2\sin^2\theta}{2kT}\right]. \tag{3}$$

Therefore, equation (2) can be reduced to a single scalar partial differential equation for Ψ

$$\Delta^*\Psi = -\frac{\mu_0^2}{2}\frac{\mathrm{d}I^2}{\mathrm{d}\Psi} - \mu_0 r^2 \sin^2\theta \exp\left[\frac{(m_\mathrm{e}+m_\mathrm{i})\Omega^2 r^2 \sin^2\theta}{2kT}\right] \times \left[\frac{\mathrm{d}g}{\mathrm{d}\Psi} + g r^2 \sin^2\theta (m_\mathrm{e}+m_\mathrm{i})\frac{\mathrm{d}}{\mathrm{d}\Psi}\left(\frac{\Omega^2}{2kT}\right)\right] \tag{4}$$

which is a version in spherical coordinates of an equation already given by Maschke and Perrin (1980) in cylindrical coordinates. When $\Omega \to 0$, $g(\Psi) \to p$ and equation (4) reduces to the Grad–Shafranov–Schlüter equation.

3. Analytical solution for the rotating case

In analogy with the corresponding static equilibrium equation, solving equation (4) requires prior knowledge of the following profiles: $g(\Psi), I(\Psi), \Omega(\Psi)$ and $T(\Psi)$. Earlier works, dealing with cylindrically symmetric systems (Missiato and Sudano 1982, Clemente and Farengo 1984) have used, for the sake of simplicity, the following assumption

$$\frac{\mathrm{d}}{\mathrm{d}\Psi}\left(\frac{\Omega^2}{2kT}\right) = 0. \tag{5}$$

We also adopt such a hypothesis and the following assumptions for g and I^2 will be made

$$I^2(\Psi) = I_0^2 \pm \frac{2\lambda^2}{\mu_0^2}\Psi^2$$
$$g(\Psi) = g_0 + \frac{\kappa^2}{\mu_0}\Psi \tag{6}$$

where, assuming Ψ positive inside the plasma and vanishing at its boundary, λ^2, κ^2, I_0^2 and g_0 are positively defined constants. The plus sign in front of λ^2 means that paramagnetic plasmas are considered, the minus sign refers to diamagnetic plasmas.

Defining a typical length R, the spherical Maschke–Perrin equation (4) can be simplified by introducing

$$x = \frac{r}{R}$$
$$\epsilon = \frac{(m_\mathrm{e}+m_\mathrm{i})\Omega^2 R^2}{3kT} \tag{7}$$

in such a way that, according to assumption (6), it takes the form

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \cot\theta\frac{\partial\Psi}{\partial\theta}\right) = -(\pm\lambda^2)R^2\Psi - \kappa^2 R^4 x^2 \sin^2\theta \exp\left(\frac{3\epsilon x^2 \sin^2\theta}{2}\right). \tag{8}$$

Expanding the exponential in the above equation in powers of ϵ, and retaining only terms up to the first order, we have

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \cot\theta\frac{\partial\Psi}{\partial\theta}\right) = -(\pm\lambda^2)R^2\Psi - \kappa^2 R^4 x^2 \sin^2\theta - \frac{3}{2}\kappa^2 R^4 \epsilon x^4 \sin^4\theta + \mathcal{O}(\epsilon^2). \tag{9}$$

Such an equation has to be considered valid in so far as $\exp\left(\frac{3\epsilon}{2}\right)$ can be well approximated by $1+\frac{3\epsilon}{2}$. Within a 4% approximation, values of ϵ up to 0.2 can be allowed. Therefore, our results will be considered appropriate only up to $\epsilon = 0.2$. This does not mean that solutions to equation (8) do not exist for larger values of ϵ, but simply that we are unable to obtain them analytically; numerical methods will be necessary in such cases. Analytical solutions for larger values of ϵ (even larger than unity) have already been found for models based in cylindrical coordinates (Maschke and Perrin 1980, Clemente and Farengo 1984). Here, the novelty is the use of spherical coordinates.

The solution of equation (9) may be written in the form

$$\Psi(x,\theta) = \Psi_{\mathrm{P}}(x,\theta) + \Psi_{\mathrm{H}}(x,\theta) \tag{10}$$

where the subscripts P and H refer to the nonhomogeneous and homogeneous solutions, respectively. A particular solution is

$$\Psi_{\mathrm{P}}(x,\theta) = Ax^2\sin^2\theta + Bx^4\sin^4\theta \tag{11}$$

where

$$\begin{aligned} A &= -\frac{\kappa^2R^2}{(\pm\lambda^2)}\left(1-\frac{12\epsilon}{(\pm\lambda^2R^2)}\right) \\ B &= -\frac{3\epsilon\kappa^2R^2}{(\pm 2\lambda^2)} \end{aligned} \tag{12}$$

and for our purposes, the homogeneous solution will be

$$\Psi_{\mathrm{H}}(x,\theta) = Cf_1^{\pm}(x)\sin^2\theta + Df_2^{\pm}(x)\sin^2\theta\left(1-\tfrac{5}{4}\sin^2\theta\right) \tag{13}$$

where C, D are coefficients to be determined by boundary conditions, and the ($\pm$) sign refers to the sign of λ^2 in (6). When the plus sign is adopted, $f_{1,2}^{+}$ are

$$\begin{aligned} f_1^{+}(x) &= \frac{\sin(\lambda Rx)}{\lambda Rx} - \cos(\lambda Rx) \\ f_2^{+}(x) &= \frac{\sin(\lambda Rx)}{\lambda Rx}\left[\frac{15}{(\lambda Rx)^2}-6\right] - \cos(\lambda Rx)\left[\frac{15}{(\lambda Rx)^2}-1\right] \end{aligned} \tag{14}$$

and when the minus sign is chosen,

$$\begin{aligned} f_1^{-}(x) &= \frac{\sinh(\lambda Rx)}{\lambda Rx} - \cosh(\lambda Rx) \\ f_2^{-}(x) &= \frac{\sinh(\lambda Rx)}{\lambda Rx}\left[\frac{15}{(\lambda Rx)^2}+6\right] - \cosh(\lambda Rx)\left[\frac{15}{(\lambda Rx)^2}+1\right]. \end{aligned} \tag{15}$$

A particular boundary condition that allows us to define C and D corresponds to the case in which the plasma is embedded by a conducting spherical shell of radius R. In this case I_0^2 must vanish and the plus sign in front of λ^2 must be chosen. Correspondingly,

$$\begin{aligned} C &= -\frac{1}{f_1^{+}(1)}\left(A+\frac{4B}{5}\right) \\ D &= \frac{4}{5}\frac{B}{f_2^{+}(1)}. \end{aligned} \tag{16}$$

The resulting model will be discussed in the following section.

4. Properties of the rotating solution

The solution derived in the previous section for a spherical boundary describes a two-parameter family of magnetic surfaces, where ϵ measures their angular velocity. Let us consider first the static limit ($\Omega = \epsilon = 0$) of this solution:

$$\Psi(x,\theta) = \frac{\kappa^2 R^2}{\lambda^2}\left[\frac{(\sin\lambda Rx/\lambda Rx) - \cos\lambda Rx}{(\sin\lambda R/\lambda R) - \cos\lambda R} - x^2\right]\sin^2\theta \tag{17}$$

This is essentially the same solution as that obtained by Morikawa (1969), although in his work he considered a thick force-free shell between a plasma sphere of radius $r = \Delta$ and the conducting shell at $r = R$. The boundary conditions one has to impose on the plasma and force-free regions constrain the possible values of the parameters λ^2 and κ^2.

In our solution, if we restrict it to cases in which Ψ has only one maximum inside the boundary, the values of λ^2 are restricted by the condition

$$0 \leqslant \sqrt{\lambda^2 R^2} \leqslant 4.493\ldots \tag{18}$$

$\lambda R = 0$ corresponds to a spherical rotating field-reversed configuration without toroidal magnetic field. $\lambda R = 4.493\ldots$ corresponds to a force-free magnetic field configuration in which the centripetal force is balanced by the pressure gradient (i.e. the pressure increases monotonously with $r\sin\theta$). In this case, care has to be taken in evaluating the constant C since κ^2 also vanishes.

In order to better illustrate these features it is convenient to write the solution at $\theta = \pi/2$:

$$\Psi(x,\pi/2) = -\left(A + \frac{4B}{5}\right)\frac{f_1^+(x)}{f_1^+(1)} - \frac{B}{5}\frac{f_2^+(x)}{f_2^+(1)} + Ax^2 + Bx^4. \tag{19}$$

Ψ has a maximum at a certain position $x = x^*$. Let us indicate this maximum as $\Psi_{\max} = \Psi(x^*,\pi/2)$ and assume it to be a fixed parameter. x^* is a function of λR and it is shown in figure 1 for different values of ϵ.

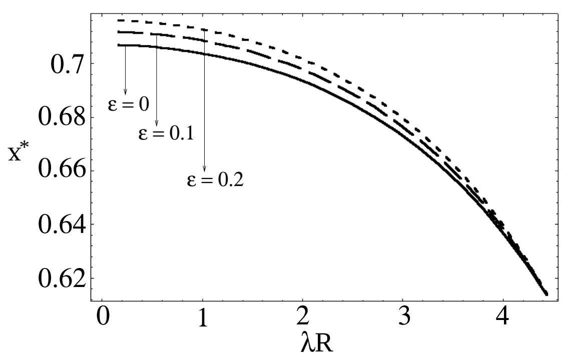

Figure 1. Magnetic axis radial location as a function of λR for different values of ϵ.

As can be seen, rotational effects are more evident for lower values of λR, resulting in an outwards shift of x^* (i.e. shifting of the magnetic axis).

The constant κ^2 appearing in the definition of $g(\Psi)$ can be related to $\Psi_{\max}$. The adimensional parameter $\frac{\kappa^2 R^4}{\Psi_{\max}}$ is a function of $x^*(\lambda R)$ and in figure 2 it has been plotted as a function of λR for different values of ϵ. As can be seen, κ^2 decreases when the rotation is increased; once again the effect is more evident at small λR.

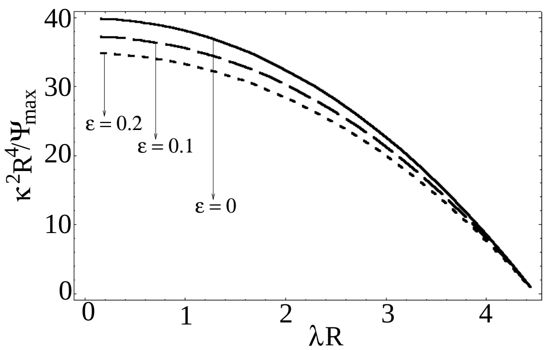

Figure 2. Adimensional pressure parameter as a function of λR for different values of ϵ.

Since the plasma pressure is $p(\Psi) = g(\Psi)\exp\left(\frac{3\epsilon x^2 \sin^2\theta}{2}\right)$, rotation is responsible for an outwards shift of constant p surfaces with relation to constant Ψ surfaces. This effect is illustrated in figure 3, where a particular case ($\lambda R = 4$, $\epsilon = 0.2$, $g_0 = 0$) is plotted.

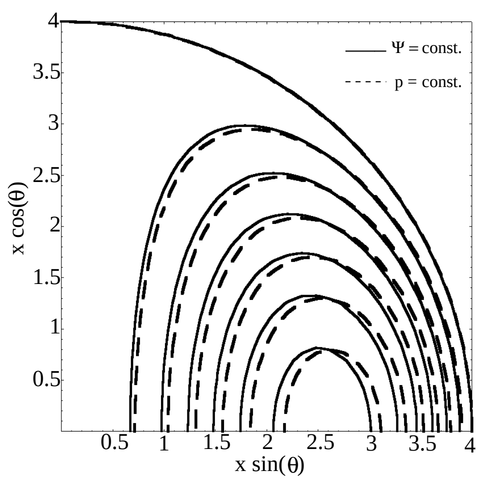

Figure 3. Magnetic flux and pressure surfaces for $\epsilon = 0.2$, $\lambda R = 4$ and $g_0 = 0$.

It is worth mentioning that the plasma average beta

$$\beta = \frac{\int \mathrm{d}V\, 2\mu_0 p}{\int \mathrm{d}V\, B^2} \tag{20}$$

is a continuous function of λR. When $g_0 = 0$ and λR tends to zero, β tends to 2.0 (Clemente and Farengo 1984); when λR tends to $4.493\ldots$, β tends to zero. So, by an appropriate choice of λR the desired β can be reproduced.

5. Conclusions

Spherically symmetric MHD equilibria with azimuthal rotation find some applications in fusion and astrophysical problems. We have obtained a pressure equilibrium equation through a procedure introduced by Maschke and Perrin (1980) for this class of geometries.

This equation is solved through the adoption of pressure and current profiles linear in the magnetic flux, a constant temperature profile, and the assumption of small rotation velocities (the rotation kinetic energy must be no greater than 20% of the thermal energy). The solution of the rotating equilibrium equation is written in a closed form, dependent on the current (λ^2) and pressure (κ^2) parameters. We have obtained solutions for both paramagnetic and diamagnetic plasmas.

The configuration studied in this paper consists of a plasmoid contained in a spherical conducting shell. In the limit of vanishing rotation the solution reduces to results obtained earlier by Morikawa (1969). For nonzero angular velocity we show the behaviour of λ^2 and κ^2 with the rotation parameter ϵ, as well as the behaviour of magnetic flux and pressure surfaces. We have restricted our analysis to an interval for λR whose limits are the first two non-negative eigenvalues of $\tan \lambda R = \lambda R$. These two limiting cases are a rotating field-reversed configuration and a force-free configuration, respectively. This latter case would be of interest for low β plasmas, as in some astrophysical situations.

An observed feature of our solution is that the magnetic axis position, located at the equatorial plane, is shifted outwards with increasing rotation velocity, and this effect is more evident for lower values of λR. Considering the Ψ value at the magnetic axis to be a constant parameter, we have observed a decrease in the pressure parameter κ^2 when rotation is increased, this effect being more pronounced for small values of λR again.

To our knowledge, this is the fourth analytic solution of the Maschke–Perrin equation in the literature. We are currently extending this solution to the configurations proposed, in the static case, by Morikawa (1969) and Morikawa and Rebhan (1970). The former includes a force-free shell between the plasma sphere and the conducting shell. The resulting boundary conditions limit the possible values for λ and κ^2. The latter configuration is surrounded by a vacuum field, and can include a magnetic dipole at the origin as well as the field produced by a straight wire at the symmetry axis.

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