Hamiltonian Representation for Magnetic Field Lines in an Exactly Soluble Model

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An exactly soluble static MHD equilibrium model with double toroidal magnetic surfaces and overall spherical symmetry is analysecl by means of a hamiltonian description for magnetic field line flow. The equilibrium Hamiltonian is perturbatively expressed in action-angle variables.

I. Introduction

One of the more important problems in magnetohydrodynamics, and not completely answerecl up to now, is the existence of solutions for the pressure equilibrium (Grad-Lust-Schluter-Shafranov) equation[1]. The relatively few known cases share as a common property some kind of spatial symmetry. A conjecture due to Grad[2] proposes the existence of solutions only for "highly symmetric" configurations. These are classified into three types - cylindrical, helical and axisymmetric toroidal[3].

In the latter class we include those MHD equilibria with overall spherical symmetry, but whose magnetic flux surfaces are multiply connected, i.e., present the topology of a torus. These solutions are often called "compact tori", and are expected to occur in two basic plasma magnetic confinement schemes - Spheromalis (low beta) and Field Reversed Configurations (high heta).

A useful tool to investigate the structure of magnetic fields in such a configuration is the study of the topology of its magnetic field lines. This is actually possihle by a Hamiltonian description - the equations for the magnetic field lines are put in a canonical form, where the role of time is played by some spatial coordinate. In an equilibrium model, the three-dimensional magnetic field line structure is equivalent to an one-degree-of-freedom Harniltonian system[4].

The purpose of this note is to show an equilibrium Hamiltonian for magnetic field lines in a compact tori moclel with overall spherical symmetry. A singular feature of this moclel is that the pressure equilihrium equation has an exact and analytical solution. Nevertheless, the exact Hamiltonian so obtained is not expressed in action-angle variables, as expected for further uses of perturbation theory. This is done perturbatively up to second order, giving an approximate Hamiltonian for the integrable case.

II. Double compact tori model

A double compact tori model with overall spherical symrnetry was proposed by Morikawa[5] with the following characteristics:

1. an interna1 spherical plasma ($0 \leq r < A$);
2. a thick spherical shell with a force-free rnagnet ($A \leq r < 1$);
3. a perfectly conducting, thin spherical shell at $r = 1$.

For this model, the pressure equilibrium equation in spherical coordinates,

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial^2\Psi}{\partial\theta^2}-\cot\theta\frac{\partial\Psi}{\partial\theta}\right)=-\mu_0 r^2\sin^2\theta\frac{dp}{d\Psi}-\mu_0^2 I\frac{dI}{d\Psi}, \tag{1}$$

admits an exact analytical solution for the transversal magnetic flux function Q, from the prior ltnowledge of the pressure p $= p(\Psi)$ and current function $\mathbf{I} = I(\Psi)$ profiles.

These are given by $p'(\Psi) = \kappa^2$ and $II'(\Psi) = \lambda^2\Psi$ inside the plasma sphere ($0 < r < A$), and $p'(\Psi) = 0$ and $II'(\Psi) = \lambda^2\Psi$ for the externa1 force-free magnet[5]. The solution of (1) consistent with these profiles and proper boundary conditions at plasma-magnet interface is

$$\Psi(r,\theta) = Af(r)\sin^2\theta, \tag{2}$$

where

$$A = \begin{cases} \kappa^2\Delta^2/\lambda^2 & (0 \le r < \Delta) \\ C_0 & (\Delta \le r < 1) \end{cases}, \tag{3}$$

$$f(r) = \begin{cases} \frac{\sin\lambda r/\lambda r) - \cos\lambda r}{\lambda\Delta/\lambda\Delta) - \cos\lambda\Delta} - \frac{r^2}{\Delta^2} & (0 \le r < \Delta) \\ (\sin\lambda_2 r/\lambda_2 r) - \cos\lambda_2 r & (A \le r < 1) \end{cases}, \tag{4}$$

and $\lambda_2 \approx 7.725$ is a positive root of $\tan\lambda = A$. In the limit $\lambda \to 0$ this solution reduces to the well-known spherical Hill's vortex. Once 9 is known, the equilibrium magnetic field components are given by

$$\begin{aligned} B_r &= -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \\ B_\theta &= \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}, \\ B_\phi &= -\frac{\mu_0 I}{r\sin\theta} = -\frac{\mu_0\lambda\Psi}{r\sin\theta}. \end{aligned} \tag{5}$$

III. Equilibrium Hamiltonian

Eq. (2) describes a family of double-toroidal nested magnetic flux surfaces, with an azimutal symmetry. In this case it is worthwhile to use a Hamiltonian representation for magnetic field lines, starting from a general formulation[6]. Salat[4] has used this technique to explore to what extent the Hamiltonian description can be used to construct magnetic fields with desired properties. It has been also applied recently to problems exhibiting cylindrical[7] as well as helical[8] symmetry.

Magnetic field line equations - in a given magnetostatic equilibrium, symmetric with respect to some (ignorable) coordinate - can be cast in a Hamiltonian form, where the role of time is played by this coordinate. The canonical equations so obtained are not actually dynamical, but the results and methods of Hamiltonian mechanics (perturbation theory, adiabatic invariance, KAM theorern, etc.) are still valid. In a general system of contravariant coordinates (x^1, x^2, x^3), let us suppose that magnetic field $\mathbf{B} = (B^1, B^2, B^3)$ is symmetric with respect to x^3. The canonically conjugated variables and "time" are[6]

$$\begin{aligned} q &= x^1, \\ p &= p(x^1, x^2, x^3) \\ &= \int dx^2\sqrt{g}B^3(x^1,x^2) + \gamma(x^1,x^3), \\ t &= x^3, \end{aligned} \tag{6}$$

where $g = \det g_{ij}$ (g_{ij} being the covariant metric tensor).

In this way, magnetic field line equations are written in a canonical form ($dq/dt = \partial H/\partial p; dp/dt = -\partial H/\partial q$), where the Hamiltonian is given by

$$H(x^1,x^2,x^3) = \int dx^2\sqrt{g}B^1(x^1,x^2) + \delta(x^1,x^3). \tag{7}$$

The functions $\gamma(x^1,x^3)$ and $\delta(x^1,x^3)$ appearing in (6) and (7) must satisfy the following constraint relation

$$\sqrt{g}B^2(x^1,x^2) + \frac{\partial H(x^1,x^2,x^3)}{\partial x^1} + \frac{\partial p(x^1,x^2,x^3)}{dx^2} = 0, \tag{8}$$

and are ultimately determined by direct comparison with the magnetic field line equations.

In order to apply these definitions to spherical geometry we assign variables as $q = $ r, t $= \phi$ and

$$p = -\mu_0 \lambda \int^{\theta} d\theta' \frac{\Psi(r,\theta')}{\sin\theta'} + \gamma(r,\phi). \qquad (9)$$

The field line Hamiltonian, according to (7), is expressed as

$$H(r,\theta,\phi) = -\Psi(r,\theta) + \delta(r,4). \qquad (10)$$

Substituting in (9) and (10) the results valid for Morikawa's model, namely eqs. (2)-(4), we have for both plasma and force-free regions

$$p(r,\theta,\phi) = \mu_0 \lambda A f(r) \cos\theta + \gamma(r,\phi), \qquad (11)$$

that can be solved for 0 in terms of p, q and t, if necessary. The Hamiltonian (10), written in terms of canonical variables, is

$$H(p,q,t) = -Af(q)\left[1 - \left(\frac{p-\gamma(q,t)}{\mu_0 \lambda A f(q)}\right)^2\right] + \delta(q,t). \qquad (12)$$

However we have assumed that t $= \phi$ is an ignorable coordinate, and this is assured if γ and δ vanish identically. So, the model Hamiltonian reduces to

$$H(p,q) = \frac{1}{\lambda^2}\frac{p^2}{f(q)} - f(q), \qquad (13)$$

where we have also made the rescalings p $\rightarrow \mu_0 A p$ and H $\rightarrow H/A$ so that p is adimensional and H is measured in units of A. Finally it turns to be convenient for further analysis to exchange coordinate and momentum - a task easily performed through a canonical transformation (p,q) $\rightarrow$ (**P**,Q), with the generating function (of the first kind) $F_1(q,$Q$) = qQ$. Noting that $f(r)$ is an even function, we rewrite (13) in the form

$$H'(P,Q) = -f(P) + \frac{1}{\lambda^2}\frac{Q^2}{f(P)}. \qquad (14)$$

IV. Perturbative obtention of action-angle variables

The Hamiltonian (14) represents an "autonomous" one-degree of freedom system (because of the ϕ-symmetry), hence it is integrable in the sense of Liouville, since H¹(P,&) itself is an integral of motion[4]. Thus it is worth making an effort to express (14) in action-angle variables, so as to allow further use of perturbation theory, like in the study of nonintegrable magnetic perturbations caused by error fields, instabilities, etc. The required transformation of variables can be done perturbatively. We express (14) in the form N(P,Q) $= H'_0(P) + \epsilon H'_1(P,Q)$, where $H'_0(P)$ is the ['unperturbed" part and H'_1(**P**,Q) is a "perturbation" (E is an order parameter which can be set equal to the unity at the end of the calculations).

Let (**P**,Q) $= (J,6)$ be the "old" action-angle variables (i.e, relative to the unperturbed system). We look for a canonical transformation (J, $\vartheta) \rightarrow (\bar{J},\bar{\vartheta})$ such that the "new" Hamiltonian H is written as a function of the "new" action $\bar{J}$ only. This can be accomplished using canonical perturbation theory[9] and the result, up to second order in E, is

$$\bar{H}(\bar{J}) = -f(\bar{J}) + \epsilon\frac{4\pi^2}{3\lambda^2}\frac{1}{f(\bar{J})} + \epsilon^2 \times$$
$$\left[-\frac{1}{18\lambda^4}\frac{16\pi^4(4\pi^2-4\pi+9)}{5}\frac{f''(\bar{J})}{f^2(\bar{J})f'^3(\bar{J})} - \frac{1}{3\lambda^4}\frac{16\pi^4(9-10\pi)}{15}\frac{1}{f^3(\bar{J})}\right]. \qquad (15)$$

The transformation equations between the "new" $(\bar{J},\bar{\vartheta})$ and "old" $(J,6)$ variables are $J = \partial S/\partial\vartheta$ and $\bar{\vartheta} = \partial S/\partial\bar{J}$, where the generating function is also expressed as a perturbation series, and reads

$$
\begin{aligned}
S(\bar{J},\vartheta) \;=\;& \bar{J}\vartheta + \epsilon\frac{1}{3\lambda^2}\vartheta(\vartheta^3 - 8\pi^3)\frac{1}{f(\bar{J})f'(\bar{J})} + \\
&+\epsilon^2\left\{-\frac{1}{18\lambda^4}\left[\frac{9\vartheta^5}{5} - 16\pi^3\vartheta^3 + \frac{16\pi^4(20\pi-9)\vartheta}{5}\right]\frac{f''(\bar{J})}{f^2(\bar{J})f'^3(\bar{J})} - \right.\\
&\left. -\frac{1}{3\lambda^4}\left[\vartheta^3 - \frac{8\pi^3(15+18\pi-20\pi^2)}{15}\right]\frac{1}{f^3(\bar{J})}\right\}, \qquad (16)
\end{aligned}
$$

so that $\bar{J}$ is expressed in terms of P aiid Q, aftci a formal inversion of series; but for practical purposes this is not really necessary. An example would be the application of secular perturbation theory to explain magnetic island formation. This task is currently being pursued and results will be published elsewhere.

V. Conclusions

Lhe magnetic field topology in an equilibrium model with double toroidal axisymmetric flux surfaces lias been studied by means of a Hamiltonian description of field lines. This is possible provided a coordinate (usually a cyclic one) plays tlie role of a time, so that tlie "dynamics" obtained is in fact tlie Lagrangean spatial structure of field lines. The model we analysed is an exact solution of pressure equilibrium equation ill spherical coordinates, aiid an exact Hamiltonian is written in terms of canonical variables. However, being an integrable system, it is important for further uses of perturbation theory to write down tlie Hamiltonian in terms of an action variable. This has been done up to second order by canonical perturbation theory and tlie averaging method.

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