Comments on the magnetic field generated by an infinite current grid

R L Viana† and I L Caldas‡

† Departamento de Física, Universidade Federal do Paraná, Caixa Postal 19081, Curitiba, 81504, Paraná, Brazil
‡ Instituto de Física, Universidade de São Paulo, Caixa Postal 20516, São Paulo, 05508, São Paulo, Brazil

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Abstract. The magnetic field generated by an infinite current grid is a useful result in Tokamak physics, but also an interesting electromagnetism problem at the undergraduate level. We outline the solution of the boundary value problem in rectangular coordinates and comment about the Hamiltonian nature of the magnetic field line equations.

Resumen. El campo magnetico producido por una grata infinita de corrientes es un resultado útil en la física de Tokamaks, pero también es un interesante problema del punto de vista didáctico. Mostramos la solución del problema de valores de contorno en coordenadas rectangulares y la naturaleza Hamiltoniana de las ecuaciones de las lineas del campo magnetico.

1. Introduction

In Tokamak confinement research, it is desirable to avoid interaction between the plasma column and the Tokamak inner wall by means of limiters (Karger and Lackner 1977, Engelhardt and Feneberg 1978, Feneberg and Wolf 1981). One such device (Belitz *et al* 1982) is a ring-shaped grid of thin current conductors, wound around the torus. The magnetic field so generated interacts resonantly with the main Tokamak equilibrium field in order to produce (at a certain critical condition) a peripheral region of stochastic magnetic field. This boundary layer is supposed to play an important role in the particle and heat plasma–wall exchange (Fuchs *et al* 1982), so the apparatus is commonly called an ergodic limiter. It has been successfully used (with a slightly more sophisticated design) in real Tokamak experiments, in part because of its robustness and simplicity (Ohyabu *et al* 1985, McCool *et al* 1989).

Martin and Taylor (1984) have proposed a theoretical model for the ergodic limiter action upon the magnetic field lines. They assume, for simplicity, a rectangular coordinate system, because: (*a*) the Tokamak is supposed to have a large aspect ratio, so it is possible to neglect toroidal effects. This approximation is often used for MHD stability analysis of Tokamaks (Bateman 1978); (*b*) the ergodic limiter action is effective only in the Tokamak boundary, minimizing the effect of the poloidal curvature. In figure 1 we show the operations which lead to this particular geometry.

In their original paper (Martin and Taylor 1984), the authors present a solution for the magnetic field generated by such a current configuration. Besides its own theoretical interest, it is a fine textbook problem for an undergraduate electromagnetism course. We have found in standard textbooks only brief mentions of this problem (Durand 1959), but through a complex variable approach, instead of the boundary value problem treatment. In the following we detail the solution, stressing the Hamiltonian nature of the corresponding magnetic field line equations.

2. Boundary value problem

The basic geometry is depicted in figure 2. The coordinates on the x axis correspond to the (rectified) poloidal circumference in the real Tokamak. Hence, the period for these coordinates is $2\pi b$ (where b is the Tokamak minor radius). On the y axis we depict the radial distance from the torus wall (the zx plane), and coordinates on the z axis stand for positions in the toroidal direction. Points with $y > 0$ are located inside the Tokamak, whereas $y < 0$ defines the external medium. As the penetration time of the metallic

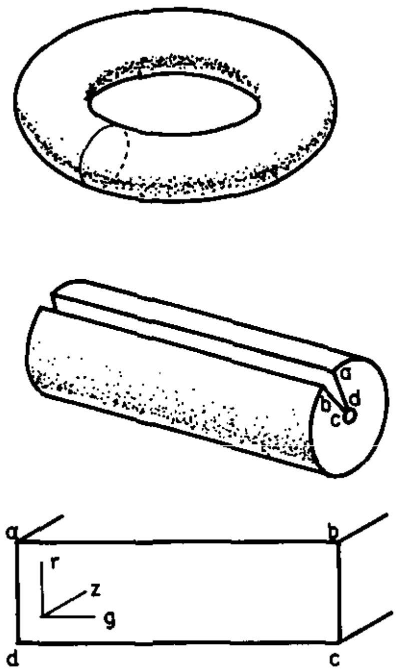


Figure 1. Operations necessary to obtain a rectangular system of coordinates for the Tokamak torus.

wall is supposed to be very small, we can ignore it and deal only with vacuum fields.

The magnetic limiter consists of a grid with m pairs of wires conducting a current I in alternate directions, such that the separation between two wires is $\pi b/m$ (see figure 3 for a sketch of the real limiter shape). In order to get a more tractable formulation of the problem, we neglect the finite extension of the grid and take it as being infinite. Outside the grid there are no further current sources, so that $\nabla \times \boldsymbol{B} = 0$ and the magnetic field can be written in terms of a magnetic scalar potential satisfying a bidimensional Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{1}$$

Using separation of variables, and considering that the effect of the current distribution must vanish far from the grid, we can write different solutions for the Tokamak (i) and external (e) regions:

$$\phi^{\mathrm{i}}(x, y) = A^{\mathrm{i}} \mathrm{e}^{-\alpha y} \sin \alpha x \tag{2a}$$

$$\phi^{\mathrm{e}}(x, y) = A^{\mathrm{e}} \mathrm{e}^{\alpha y} \sin \alpha x \tag{2b}$$

The single-valuedness of both solutions in the poloidal direction leads to $\alpha = N/b$, where N is a positive integer, so that the general solution is a superposition of harmonics. The magnetic field components (for $y < 0$ and $y > 0$) are:

$$B_x^{\mathrm{i,e}} = \sum_{N=0}^{\infty} \frac{N}{b} A_N^{\mathrm{i,e}} \mathrm{e}^{\pm Ny/b} \cos\left(\frac{Nx}{b}\right) \tag{3a}$$

$$B_y^{\mathrm{i,e}} = \pm \sum_{N=0}^{\infty} \frac{N}{b} A_N^{\mathrm{i,e}} \mathrm{e}^{\pm Ny/b} \sin\left(\frac{Nx}{b}\right). \tag{3b}$$

Imposing the continuity of the normal (y) component across the interface (namely the plane $y = 0$) we have:

$$A_N^{\mathrm{e}} = -A_N^{\mathrm{i}}. \tag{4}$$

The second boundary condition involves the superficial current density $\boldsymbol{j}_{\mathrm{s}} = j_{\mathrm{s}} \boldsymbol{e}_z$, and reads:

$$\boldsymbol{e}_y \times (\boldsymbol{B}^{\mathrm{i}} - \boldsymbol{B}^{\mathrm{e}})_{y=0} = \mu_0 \boldsymbol{j}_{\mathrm{s}} \tag{5}$$

where the current density on the grid can be simulated by a sequence of delta functions:

$$j_{\mathrm{s}} = I \sum_{k=1}^{2m} (-1)^k \, \delta\left(x - \frac{\pi b}{m} k\right). \tag{6}$$

Developing the delta function in a Fourier series, we pick up the Nth order harmonic of this distribution:

$$j_{\mathrm{s},N} = \frac{I}{\pi b} \sum_{k=1}^{2m} (-1)^k \cos\left[N\left(\frac{x}{b} - \frac{\pi}{m} k\right)\right]. \tag{7}$$

The summation above is found in standard tables (as in Jolley 1965). It gives $2m \cos(Nx/b)$ where N is equal to $(2p+1)m$ (where p is a positive integer), and vanishes otherwise. Hence the current harmonics become equal to:

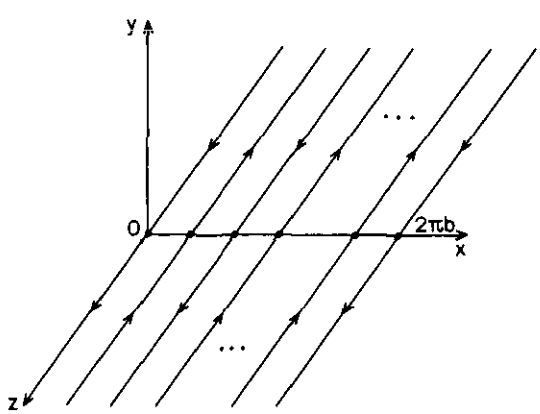
$$j_{\mathrm{s},N} = \frac{2mI}{\pi b} \cos\left(\frac{Nx}{b}\right) \delta_{N,(2p+1)m} \tag{8}$$

The application of equation (5) to field and current harmonics determines the unknown constants in (3):

$$A_{N,p}^{\mathrm{i}} = -\frac{\mu_0 I}{\pi(2p+1)}. \tag{9}$$

Inside the Tokamak chamber, the magnetic field com-

Figure 2. Basic geometry of the current grid.



ponents are given by a sequence of p modes:

$$B^{i}_{x,p} = -\frac{\mu_0 m I}{\pi b}\exp\left(\frac{-(2p+1)my}{b}\right)\cos\left(\frac{(2p+1)mx}{b}\right) \tag{10a}$$

$$B^{i}_{y,p} = \frac{\mu_0 m I}{\pi b}\exp\left(\frac{-(2p+1)my}{b}\right)\sin\left(\frac{(2p+1)mx}{b}\right). \tag{10b}$$

However, mainly due to the exponential dependence in (10), the $p = 0$ mode has a dominant effect upon the other ones, and is usually the only term to be retained in the calculations. It is often called pure ion mode:

$$B_{x,y} = \pm\frac{\mu_0 m I}{b\pi}\,e^{-my/b}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(\frac{mx}{b}\right). \tag{11}$$

3. Hamiltonian approach

We adopt the so-called cylindrical model for the Tokamak (Filonenko *et al* 1967), so that the limiter field given by (11) is superposed to a uniform toroidal field in the z direction: $\boldsymbol{B}_z = B_0\boldsymbol{e}_z$. The magnetic field line equations are thus:

$$dx/B_x = dy/B_y = dz/B_0. \tag{12}$$

The idea of treating this set of equations as canonical equations from a Hamiltonian is not new. Kerst (1962) proposed as canonical variables the following coordinates (in a rectangular geometry):

$$q = x \tag{13a}$$

$$p = \int^{y} B_z\,dy = B_0 y \tag{13b}$$

z being the (rescaled) timelike variable. In this case one can rewrite (12) in a canonical form:

$$dx/dz = d\mathscr{H}/\partial p \tag{14a}$$

$$dp/dz = -\partial\mathscr{H}/\partial x \tag{14b}$$

where $\mathscr{H} = \mathscr{H}(x,p)$ is the field line flow Hamiltonian.

Comparing (12) and (14) we get the following set of equations:

$$\partial\mathscr{H}/\partial p = B_x \qquad \partial\mathscr{H}/\partial x = -B_y \tag{15}$$

which are satisfied by the Hamiltonian:

$$\mathscr{H} = \frac{\mu_0 I}{\pi}\exp\left(\frac{-my}{b}\right)\cos\left(\frac{mx}{b}\right) \tag{16}$$

in the case of equation (11).

As the number of degrees of freedom matches the number of integrals of motion (actually there is only one, namely the Hamiltonian), this system is said to be integrable, in the Liouville sense. However, in the real Tokamak case, the current grid has a finite extension, supposed to be small when compared with the torus dimensions. So, even neglecting border effects, one is faced with an explicit z dependence in the Hamiltonian, which destroys the integrability property and allows a more complicated dynamics, where it is possible (and even desirable, as in the case of the ergodic limiter) to find chaotic behaviour in the magnetic field line flow (Martin and Taylor 1984).

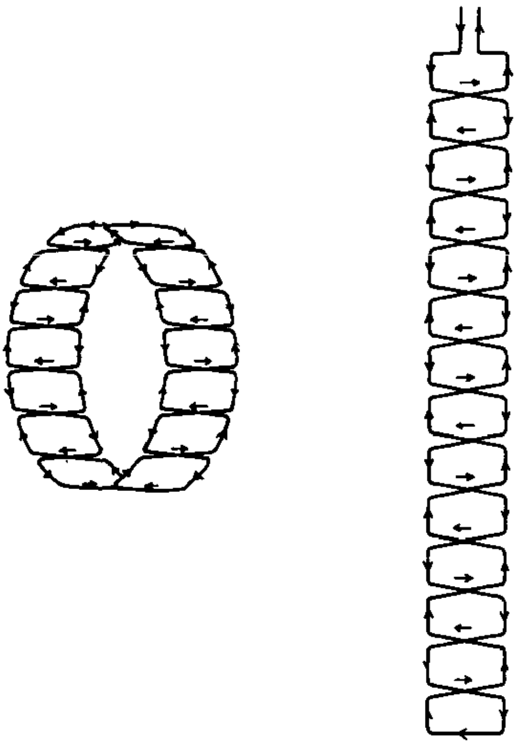


Figure 3. Current grid design for a Tokamak experiment.

This fact suggests that it is possible to use, at least in principle, the Hamiltonian (16) as a finite extension perturbation, acting upon an equilibrium Tokamak Hamiltonian $\mathscr{H}_0$. The latter invariant is already known in the plasma literature (Fernandes *et al* 1988). The standard perturbation theory is applicable to this combination, and quantitative predictions about the magnetic field line dynamics could be available. We are pursuing this task and results will be published elsewhere (Viana and Caldas 1991).

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