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ANALYTIC STOCHASTIC REGULARIZATION IN QCD AND ITS SUPERSYMMETRIC EXTENSION*

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We outline some features of stochastic quantization and regularization of fermionic fields with applications to spinor QCD, showing the appearance of a non-gauge invariant counterterm. We also show that non-invariant terms cancel in supersymmetric multiplets.

1. One of the most remarkable features of Parisi-Wu's stochastic quantization method[1] is the possibility of an original prescription for regularization, built upon a non-Markovian extension of the actual process. By means of a particular regulator function, as shown by Alfaro,[2] it is possible to obtain a scheme very similar to Speer's analytic regularization.[3] At the beginning, the commonly accepted idea was that this new method could respect all physical invariances, in particular, gauge symmetry.[4] However, a one-loop computation of counterterms in 4-dimensional scalar Yang-Mills theory, shows the existence of a non-gauge invariant counterterm.[5] We shall follow the same steps and use the analytic stochastic regulator in 4-dimensional spinorial QCD.[6] As a byproduct of our calculations for spinor and scalar cases, we verify that a gauge field coupled to a supersymmetric matter multiplet receives only gauge invariant contributions to the counterterm. Therefore, at least to lowest order, the regularization scheme preserves gauge invariance and supersymmetry.

2. Originally developed for bosonic models, it was only recently that stochastic quantization of fermions received a physically consistent treatment.[7] The starting point is a generalization of the original Langevin equation by means of the introduction of a kernel $k_{ij}(x,y)$

$$\frac{\partial \varphi_i}{\partial \tau}(x,\tau) = -\int d^D y\, k_{ij}(x,y)\frac{\delta S[\varphi]}{\delta \varphi_j(y,\tau)} + \eta_i(x,\tau)\,, \tag{1}$$

where $S[\varphi]$ is the Euclidean action, and $\eta(x,\tau)$ a classical noise field, with

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correlations

$$\langle \eta_i(x,\tau) \rangle^{\text{conn}} = 0$$
$$\langle \eta_i(x,\tau)\,\eta_j(x',\tau') \rangle = 2k_{ij}(x,x')\delta(\tau-\tau')\,. \tag{2}$$

In the case of free fermions with the classical Euclidean action[a]

$$S[\bar{\psi},\psi] = -i\int d^4x\,\bar{\psi}(x)(\partial\!\!\!/ + im)\psi(x)\,, \tag{3}$$

the kernel is given by

$$k^F_{\alpha\beta}(x,y) = (i\partial\!\!\!/_x + m)_{\alpha\beta}\delta^4(x-y) \tag{4}$$

and the Langevin equation is

$$\frac{\partial\psi}{\partial\tau} = (\partial^2 + m^2)\psi + \vartheta\,. \tag{5}$$

The effect of the kernel is a Langevin equation of exactly the same type as that obeyed by the bosonic fields. The Green function (uncrossed propagator) is given in momentum space by

$$G_F(k;\tau-\tau')_{\alpha\beta} = \delta_{\alpha\beta}e^{-(k^2+m^2)(\tau-\tau')}\vartheta(\tau-\tau')\,. \tag{6}$$

3. The four-dimensional action for Euclidean QCD is given by

$$S[A,\psi,\bar{\psi}] = \int d^4x\left[\frac{1}{4}(F^a_{\mu\nu})^2 - i\bar{\psi}(D\!\!\!\!/ + im)\psi\right]. \tag{7}$$

As proposed by Ishikawa,[8] we should use modified covariant kernels

$$k_{\alpha\beta}(x,y) = i(D\!\!\!\!/_x - im)_{\alpha\beta}\delta^4(x-y)\,. \tag{8}$$

In the one-loop computation we shall perform, the linearized kernel Eq. (4) suffices for the noise (Eq. (2)).

[a]Our Euclidean γ-matrices satisfy $\{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu}$.

Thus, the following Langevin equations hold,

$$\dot{\psi}_\alpha = -(\not{D} - im)(\not{D} + im)_{\alpha\beta}\psi_\beta + \vartheta_\alpha \tag{9a}$$

$$\dot{\bar{\psi}}_\alpha = -\bar{\psi}_\beta(\not{D}' - im)^T(\not{D}' - im)^T_{\beta\alpha} + \bar{\vartheta}_\alpha\,, \tag{9b}$$

where $D'_\mu = -\partial_\mu - ieA_\mu$, and for the gauge field,

$$\dot{A}_\mu = -\partial_\nu F_{\mu\nu} + e\bar{\psi}\gamma_\mu\psi + \eta_\mu\,. \tag{9c}$$

Stochastic regularization of ultraviolet divergences requires the introduction of a non-Markovian noise, smearing the fictitious time delta function. In momentum space

$$\langle\vartheta_\alpha(k,\tau)\bar{\vartheta}_\beta(k',\tau')\rangle = (-\not{k} + m)_{\alpha\beta}\delta^4(k + k')f_\varepsilon(\tau - \tau') \tag{10a}$$

$$\langle\eta_\mu(k,\tau)\eta_\nu(k',\tau')\rangle = \delta_{\mu\nu}\delta^4(k + k')f_\varepsilon(\tau - \tau')\,, \tag{10b}$$

where $f_\varepsilon(\tau)$ is a regulator function, such that

$$\lim_{\varepsilon\to 0} f_\varepsilon(\tau) = 2\delta(\tau)\,. \tag{11a}$$

We currently use the representation

$$f_\varepsilon(\tau) = \varepsilon|\tau|^{\varepsilon - 1} \tag{11b}$$

and get in Fourier space

$$\tilde{f}_\varepsilon(\omega) = \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}f_\varepsilon(\tau)e^{-i\omega\tau} = 2\hat{f}_\varepsilon|\omega|^{-\varepsilon}\,, \tag{11c}$$

with

$$\hat{f}_\varepsilon \equiv \varepsilon\Gamma(\varepsilon)\sin\frac{\pi}{2}(1 - \varepsilon)\,. \tag{11d}$$

The two point fermionic correlation function (crossed propagator) at lowest order (namely, with linearized kernel for the noise) is given by

$$\Delta_{\alpha\beta}(k;\tau,\tau') = \langle \psi_\alpha(k,\tau), \bar{\psi}_\beta(-k,\tau')\rangle$$

$$= 2\int_0^\tau d\tau'' \int_0^\tau d\tau'''\, G_{\alpha\gamma}(k,\tau-\tau'')\,G_{\gamma\beta}(-k,\tau'-\tau''')\,f_\varepsilon(\tau''-\tau'''), \quad (12)$$

which, after using (6) and (11), integrates to

$$\Delta_{\alpha\beta}(k;\tau,\tau') = \hat{f}_\varepsilon \frac{(\not{k}-m)_{\alpha\beta}}{(k^2+m^2)^{1+\epsilon}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \frac{e^{-ix(k^2+m^2)(\tau-\tau')}|x|^{-\epsilon}}{1+x^2}. \quad (13)$$

A virtue of the regulator (11b) is that one is led to meromorphic amplitudes, and the ultraviolet divergences show up as poles in the ε parameter, as is the case in analytic regularized expressions.[3] In expression (13), we take only the first few terms in the ϵ expansion. In order to complete the Feynman rules,[9] we compute the gauge field crossed propagator

$$D_{\mu\nu}(k;\tau,\tau') = \hat{f}_\varepsilon \frac{\delta_{\mu\nu}}{(k^2)^{1+\epsilon}} \int \frac{dx}{\pi} \frac{e^{-ixk^2(\tau-\tau')}}{1+x^2}|x|^{-\epsilon}, \quad (14)$$

and the vertices are the same as is usual in perturbation theory.

4. We shall compute the gauge field counterterm due to matter fields internal lines. The relevant divergent diagrams are (see (5))
i) the two point function with 2 internal crossed lines — Fig. (1a);
ii) the two point function with one internal crossed line — Fig. (1b);
iii) the three point function with one internal crossed line — Fig. (1c);
iv) the four point function with one internal crossed line — Fig. (1d).

For the divergent part, we can use $|x|^{-\epsilon} = 1$ in (13). Diagram G_1 in Fig. (1a) contributes as follows

$$G_1 = \frac{2}{p^2}\int \frac{d^4k}{(2\pi)^4} \frac{-\delta_{\mu\nu}(k^2+m^2+k.p) + 2k_\mu k_\nu + k_\mu p_\nu + k_\nu p_\mu}{[(p+k)^2+m^2]^{1+\epsilon}(k^2+m^2)^{1+\epsilon}(p^2+k^2+k.p.+m^2)}, \quad (15)$$

which gives, for the divergent piece,[b] after expanding the integrand in powers of the external momenta in order to isolate the pole (further terms in this expansion are finite)

[b] We use the simple formula $\int d^Dk/(2\pi)^D\ ((k^2)^n/(k^2+1)^{m+\varepsilon}) = 1/(2\sqrt{\pi})^D\ (\Gamma(D/2+n) \times \Gamma(m+\varepsilon+D/2-n)/\Gamma(D/2)\Gamma(m+\varepsilon))$.

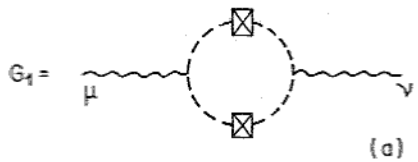

Fig. 1a. QED 2-point function with two internal crossed lines

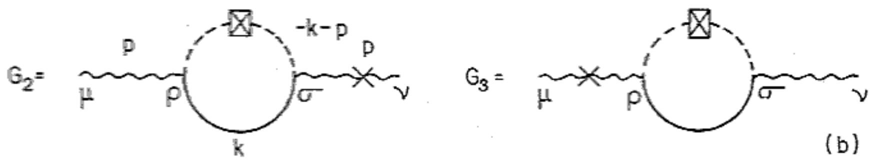

Fig. 1b. 2-point function with an external crossed line.

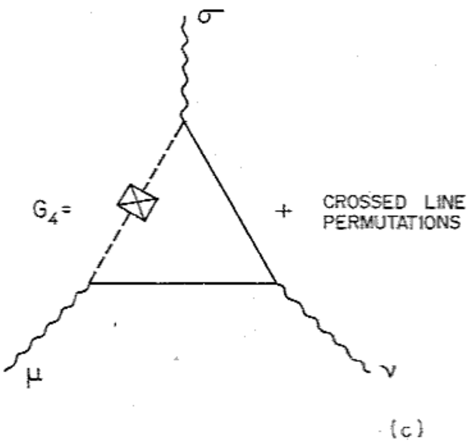

Fig. 1c. 3-point function with an internal crossed line.

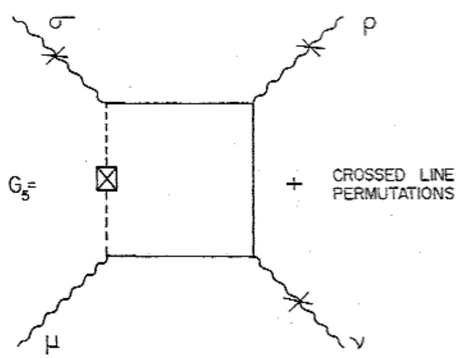

Fig. 1d. 4-point function with an internal crossed line.

$$G_1 = -\frac{\delta_{\mu\nu}}{(32p^2\pi^2\epsilon)}, \tag{16}$$

which is
i) half the value obtained in dimensional regularization with $D = 4 - 2\varepsilon$;
ii) minus twice the corresponding bosonic value.

For diagrams G_2 and G_3 in Fig. (1b), we have

$$G_2 + G_3 = 4\int_{-\infty}^{\tau} dt_2 \int_{-\infty}^{t_2} dt_1 \int \frac{d^4k}{(2\pi)^4} G_{\mu\rho}(p;\tau,t_1) D_{\sigma\nu}(p;\tau,t_2)$$
$$\times \operatorname{tr}\gamma_\rho \Delta(k+p;t_1,t_2)\gamma_0(-\not k + m) G(k;t_1,t_2) . \tag{17}$$

The result for the divergent piece is

$$G_2 + G_3 = \frac{7\delta_{\mu\nu}}{48p^2\pi^2\varepsilon} - \frac{p_\mu p_\nu}{12\pi^2(p^2)^2\varepsilon}, \tag{18}$$

which is equal to the corresponding dimensionally regularized result with $D = 4 - 2\varepsilon$. For the polarization tensor, we have (divergent piece)

$$\pi_{\mu\nu}^{\text{div}}(p) = \frac{11p^2\delta_{\mu\nu}}{96\pi^2\varepsilon} - \frac{p_\mu p_\nu}{12\pi^2\varepsilon}, \tag{19}$$

which is non-transverse due to the double crossed diagram, which is half the necessary value (notice that $2G_1 + G_2 + G_3$ is tranverse). If we compute the remaining diagrams, we have

$$G_4 = -\frac{i}{2\pi^2\varepsilon}(p_\nu\delta_{\mu\sigma} - p_\mu\delta_{\nu\sigma}) \tag{20}$$

for the 3-point function, and

$$G_5 = \frac{1}{\pi^2\varepsilon}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}) \tag{21}$$

for the 4-point function (always the divergent piece).

The counterterm reads

$$\delta L = -\frac{1}{48\pi^2\varepsilon}(F^a_{\mu\nu})^2 - \frac{1}{128\pi^2\varepsilon}A_\mu\partial^2 A^\mu . \tag{22}$$

If the gauge field is coupled to a supersymmetric matter multiplet (2 bosonic charged fields and one Dirac field) the problematic term cancels, since

$$(G_1)^{Bos} = \frac{\delta_{\mu\nu}}{64\pi^2\varepsilon} p^2 \tag{23a}$$

$$(G_2 + G_3 + G_3')^{Bos} = -\frac{5\delta_{\mu\nu} p^2}{6(4\pi)^2\varepsilon} + \frac{p_\mu p_\nu}{3(4\pi)^2\varepsilon} \tag{23b}$$

$$(G_4)^{Bos} = +\frac{i}{8\pi^2\varepsilon}(p_\nu\delta_{\mu\sigma} - p_\mu\delta_{\nu\sigma}) \tag{23c}$$

$$(G_5)^{Bos} = -\frac{1}{4\pi^2\varepsilon}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}) \tag{23d}$$

and the corresponding counterterm is

$$\delta_{Bos} L = \frac{1}{12(4\pi)^2\varepsilon}(F^a_{\mu\nu})^2 + \frac{1}{256\pi^2\varepsilon} A_\mu \partial^2 A^\mu . \tag{24}$$

For the SUSY matter multiplet, we have

$$\delta_{SUSY} L = \delta_F L = 2\delta_B L = -\frac{1}{96\pi^2\varepsilon}(F^a_{\mu\nu})^2 , \tag{25}$$

which is gauge invariant.

In the case of $N = 1$ SUSY Yang-Mills theory, the result is similar. We have one Majorana fermion in the adjoint representation contributing, in the case of the gauge two point function as in Fig. (2a), and the Yang-Mills self-interaction

$$L_{int} = \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)[A_\mu, A_\nu]^a + \frac{1}{4}[A_\mu, A_\nu]^{a^2} ,$$

which gives the same contribution as scalar matter as in Fig. (2b) but for a factor of 2 coming from a combinatorial factor (as in Wick theorem, in perturbation theory), providing a gauge invariant result, for the counterterm

$$\delta L = -\frac{1}{96\pi^2\varepsilon}(F^a_{\mu\nu})^2 . \tag{26}$$

5. As far as supersymmetry is concerned, there is no breaking originating from

Fig. 2a. Yang-Mills two-point self-interaction.

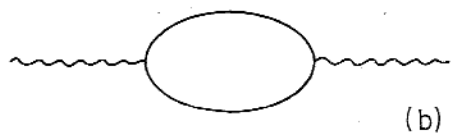

Fig. 2b. Yang-Mills-Matter two-point function.

the regularization which is independent of space-time, regularizing fermions and bosons in the same way. Notice also that supersymmetric cancellation of divergences cancel as usual.

It remains to be verified that in supersymmetric models the gauge invariance principle is maintained valid in higher loop order, and for finite terms.

Gauge fixing has been discussed in detail in the first reference[5] in the framework of stochastic quantization. In the present paper, we are dealing with one loop contributions from the matter fields. The Yang-Mills self-interactions give an independent contribution: one could, as an example, work with several flavors, and any non-gauge invariance should cancel between matter fields themselves,[5b] as well as among gauge fields and Fadeev-Popov ghosts.[5a]

In the abelian case, the problem is not relevant, consisting only of a trivial extra gauge field interaction $(\partial_\mu A^\mu)^2$ which is of the type of a usual gauge fixing, while in the non-abelian theory, we have a modification of the physical modes, in a non-gauge invariant way. It is the coefficient of this non-invariant term that vanishes in supersymmetric theories.

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