GAS CIRCULATION DUE TO AN AZIMUTHAL TEMPERATURE DISTRIBUTION OVER A MICRO-TUBE WALL

Felix Sharipov
Departamento de Física, Universidade Federal do Paraná
Caixa Postal 19044, Curitiba, Paraná, 81531-990 Brazil
Email: sharipov@fisica.ufpr.br

ABSTRACT

In many applications, a temperature of fixed cross section of micro-tube is not uniform, but it can deviate from its equilibrium value. In this case the gas flow is not axi-symmetrical any more, but the flow is two dimensional. Under such conditions, the density cannot be considered as uniform across the section and gas circulations can occur. The aim of the present work is numerical calculation of distributions of velocity, density and temperature caused by an azimuthal temperature distribution over the tube wall. The problem is solved on the basis of the kinetic equation over a wide range of the gas rarefaction.

NOMENCLATURE

c dimensionless molecular velocity
f velocity distribution function
f* Maxwellian distribution function
h perturbation function
k Boltzmann constant
m molecular mass
n local number density
n₀ equilibrium number density
P local pressure
P₀ equilibrium pressure
q dimensionless heat flux vector
q' heat flux vector
R radius of tube
r position vector
T local temperature
T₀ equilibrium temperature
T_w wall temperature
u max maximum bulk velocity
u dimensionless bulk velocity
u' bulk velocity
V peculiar molecular velocity
v molecular velocity
v₀ most probable molecular speed
W weight function
δ rarefaction parameter
θ azimuthal angle in velocity space
μ viscosity
ρ density deviation
ρ max maximum density deviation
ρ min minimum density deviation
τ temperature deviation
φ perturbation function
φ azimuthal angle in physical space
ψ perturbation function

1 Statement of the problem

When the temperature of a cross section of micro-tube is not uniform, the gas flow is not axisymmetrical any more. In this case a gas circulations occurs due to the thermal creep on the tube wall. Such phenomena are well known and some examples are given in Chapter 5 of Ref. [1]. Here, we will consider a circular cross section, which is most typical in practice.
Consider a gas confined inside of a long tube with an azimuthal temperature distribution \( T_w(\varphi) \). Since the longitudinal pressure and temperature gradients are absent, the gas does not flow along the tube, but it can circulate within the cross section.

In our calculations, we assume the following azimuthal distribution over the tube wall

\[
T_w(\varphi) = T_0 + \Delta T \sin^2 \varphi, \tag{1}
\]

where \( T_0 \) is the equilibrium temperature, \( 0 \leq \varphi \leq 2\pi \) is the azimuthal variable shown in Fig.1 and \( \Delta T \) is the maximum temperature deviation which is small compared with the equilibrium temperature, i.e.

\[
|\Delta T| \ll T_0. \tag{2}
\]

For a given temperature distribution the main parameter determining the solution is the gas rarefaction defined as

\[
\delta = \frac{P_0 R}{\mu v_0}, \tag{3}
\]

where \( P_0 \) is the equilibrium pressure, \( R \) is the tube radius, \( \mu \) is the gas viscosity, \( v_0 \) is the most probable molecular speed given by

\[
v_0 = \left( \frac{2kT_0}{m} \right)^{1/2}, \tag{4}
\]

\( k \) is the Boltzmann constant, \( m \) is the molecular mass.

2 Input equation

For our aim the S-model [2] of the Boltzmann equation, which gives the correct expression for the viscosity and the thermal conductivity in the hydrodynamic regime, is most suitable. This model equation reads

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \frac{P}{\mu} \times \left\{ f^M \left[ 1 + \frac{2m}{15P k T} \mathbf{q}' \cdot \mathbf{v} \left( \frac{mv^2}{2kT} - \frac{5}{2} \right) \right] - f \right\}, \tag{9}
\]

where \( f = f(t, \mathbf{r}', \mathbf{v}) \) is the distribution function, \( t \) is the time, \( \mathbf{r}' \) is the position vector, \( \mathbf{v} \) is the molecular velocity, \( P \) is the local pressure, \( f^M \) is the local Maxwellian defined as

\[
f^M = n \frac{m}{2\pi k T} \left( \frac{m v^2}{2kT} \right)^{3/2} \exp \left( -\frac{m v^2}{2kT} \right), \tag{10}
\]

We are going to calculate the following quantities:

Relative density deviation

\[
\rho(y, z) = \frac{n(y, z) - n_0}{n_0} \frac{T_0}{\Delta T}, \tag{5}
\]

where \( n(y, z) \) is the local number density.

Relative temperature deviation

\[
\tau(y, z) = \frac{T(y, z) - T_0}{\Delta T}, \tag{6}
\]

where \( T(y, z) \) is the local temperature.

Dimensionless bulk velocity

\[
u_y = \frac{u'_y T_0}{v_0 \Delta T}, \quad u_z = \frac{u'_z T_0}{v_0 \Delta T}, \tag{7}
\]

where \( \mathbf{u}' = \{u'_y, u'_z\} \) is the dimensional bulk velocity.

Dimensionless heat flux

\[
q_y = \frac{q'_y P_0}{v_0 \Delta T}, \quad q_z = \frac{q'_z P_0}{v_0 \Delta T}, \tag{8}
\]

where \( \mathbf{q}' = \{q'_y, q'_z\} \) is the dimensional heat flux.
and $\mathbf{V}$ is the peculiar molecular velocity

$$\mathbf{V} = \mathbf{v} - \mathbf{u}' .$$ \hspace{1cm} (11)

The local values of the number density $n$, the temperature $T$, the bulk velocity $\mathbf{u}'$ and the heat flow vector $\mathbf{q}'$ are calculated as

$$n(t, r') = \int f(t, r', \mathbf{v}) \, d\mathbf{v} ,$$ \hspace{1cm} (12)

$$T(t, r') = \frac{m}{3nk} \int f(t, r', \mathbf{v}) V^2 \, d\mathbf{v} ,$$ \hspace{1cm} (13)

$$\mathbf{u}'(t, r') = \frac{1}{n} \int f(t, r', \mathbf{v}) \mathbf{v} \, d\mathbf{v} ,$$ \hspace{1cm} (14)

$$\mathbf{q}'(t, r') = \frac{m}{2} \int f(t, r', \mathbf{v}) V^2 \mathbf{V} \, d\mathbf{v} .$$ \hspace{1cm} (15)

Here, we consider the steady-state gas flow, therefore the dependence of the distribution function and its moments on the time will be omitted.

Since $\Delta T$ is significantly smaller than the equilibrium temperature $T_0$, the kinetic equation can be linearized with respect to $\Delta T / T_0$ so that

$$f(r, \mathbf{c}) = f^0 \left[ 1 + h(r, \mathbf{c}) \frac{\Delta T}{T_0} \right] ,$$ \hspace{1cm} (16)

where $f^0$ is the absolute Maxwellian corresponding to the equilibrium state at the number density $n_0$ and temperature $T_0$

$$f^0 = \frac{n_0}{(\sqrt{\pi} v_0)^3} \exp(-c^2) ,$$

$r$ and $\mathbf{c}$ are dimensionless variables

$$\mathbf{r} = \frac{r'}{R} , \quad \mathbf{c} = \frac{\mathbf{v}}{v_0} .$$

Substituting (16) into (9), (12)-(15) and taking into account (2) - (8) we obtain the linearized kinetic equation

$$\mathbf{c} \cdot \frac{\partial h}{\partial \mathbf{r}} = \delta \left\{ \mathbf{v} + 2 \mathbf{c} \cdot \mathbf{u} + \tau \left( c^2 - \frac{3}{2} \right) \right\} + \frac{4}{15} \mathbf{c} \cdot \mathbf{q} \left( c^2 - \frac{5}{2} \right) - h \right\} .$$ \hspace{1cm} (17)

The moments introduced above are expressed in term of the perturbation function as

$$v = \frac{1}{\sqrt{\pi} \tau} \int \exp(-c^2) h \, dc ,$$ \hspace{1cm} (18)

$$\tau = \frac{1}{\sqrt{\pi} \tau} \int \exp(-c^2) h \left( \frac{2}{3} c^2 - 1 \right) \, dc ,$$ \hspace{1cm} (19)

$$u = \frac{1}{\sqrt{3 \pi} \tau} \int \exp(-c^2) h \, dc ,$$ \hspace{1cm} (20)

$$q = \frac{1}{\sqrt{3 \pi} \tau} \int \exp(-c^2) h \left( c^2 - \frac{5}{2} \right) \, dc .$$ \hspace{1cm} (21)

The diffuse gas-surface interaction is assumed on the tube wall, i.e.

$$h = v_w + \left( c^2 - \frac{3}{2} \right) \sin^2 \phi \quad \text{at} \quad r = 1, \ c_r \leq 0 ,$$ \hspace{1cm} (22)

where $c_r$ is the radial molecular velocity and the quantity $v_w$ is calculated from the condition of wall impermeability. In other words, the normal component of the bulk velocity must be zero on the tube wall.

3 Numerical scheme

Thus, we have the integral-differential equation given by Eqs.(17)-(21) with the boundary condition (22). The equation is solved by the discrete velocity method [3–7].

In order to eliminate the variable $c_r$, new perturbation functions are introduced as

$$\phi(y, z, c_y, c_z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-c_r^2) h(y, z, c_y, c_z) \, dc_r$$ \hspace{1cm} (23)

$$\psi(y, z, c_y, c_z) = \frac{1}{\sqrt{\pi} \tau} \int_{-\infty}^{\infty} \exp(-c_r^2) \left( c_r^2 - \frac{1}{2} \right)$$
Then Eq.(17) is substituted by the following two equations

\[ \mathbf{c} \cdot \frac{\partial \phi}{\partial \mathbf{r}} = \delta \left[ \nu + 2 \mathbf{c} \cdot \mathbf{u} + \tau (c_p^2 - 1) \right] + \frac{4}{15} \mathbf{c} \cdot \mathbf{q} (c_p^2 - 2) - \phi \]  

(25)

\[ \mathbf{c} \cdot \frac{\partial \psi}{\partial \mathbf{r}} = \delta \left[ \frac{1}{2} \tau + \frac{2}{15} \mathbf{c} \cdot \mathbf{q} - \psi \right]. \]  

(26)

These equations are coupled via the moments calculated as

\[ \nu = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) \phi c_p \, dc_p \, d\theta, \]  

(27)

\[ \tau = \frac{2}{3\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) [\phi (c_p^2 - 1) + \psi] c_p \, dc_p \, d\theta, \]  

(28)

\[ u_y = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) \phi c_p^2 \cos \theta \, dc_p \, d\theta, \]  

(29)

\[ u_z = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) \phi c_p^2 \sin \theta \, dc_p \, d\theta, \]  

(30)

\[ q_y = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) [\phi (c_p^2 - 2) + \psi] c_p^2 \cos \theta \, dc_p \, d\theta, \]  

(31)

\[ q_z = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-c_p^2) [\phi (c_p^2 - 2) + \psi] c_p^2 \sin \theta \, dc_p \, d\theta. \]  

(32)

Here, the Cartesian coordinates in the velocity space \((c_y, c_z)\) have been replaced by the polar coordinates \((c_p, \theta)\), i.e.,

\[ c_y = c_p \cos \theta, \quad c_z = c_p \sin \theta. \]  

(33)

According to the discrete velocity method a grid in the velocity space \((c_p, \theta)\) is introduced. For the variable \(c_p\) the Gaussian abscissas corresponding to the weight function

\[ W(c_p) = \frac{1}{\pi} \exp \left( -c_p^2 \right) \]  

(34)

are used, while a regular grid is introduced for the variable \(\theta\).

In the physical space a non-regular grid is introduced in \(y\) and \(z\) directions in order to obtain the grid lines intersections on the circle boundary as is shown in Fig. 2. Let us note \(\alpha_i\) the angle characterizing the grid cells so that \(\tan \alpha_i = \Delta y_j / \Delta x_i\) as is shown in Fig. 3. The explicit finite difference scheme for the function \(\phi\) at the fixed point \((c_p, \theta)\) of the velocity space has the following form (see Fig. 3):

\[ \phi_{ij}(c_p, \theta) = \frac{\delta \Phi_{ij} + \frac{c_p}{\alpha} \Phi_b(c_p, \theta)}{\delta + \frac{c_p}{\alpha}}, \]  

(35)

where

\[ \phi_{ij}(c_p, \theta) = \phi(x_i, y_j, c_p, \theta), \]  

(36)

\[ \Phi_{ij} = u(y_i, z_j) + \tau(y_i, z_j) (c_p^2 - 1) + 2 \left[ u_x(y_i, z_j) \cos \theta + u_z(y_i, z_j) \sin \theta \right] c_p \]  

\[ + \frac{4}{15} \left[ q_y(y_i, z_j) \cos \theta + q_z(y_i, z_j) \sin \theta \right] c_p (c_p^2 - 2) \]  

(37)

\[ \Delta x = \left\{ \begin{array}{ll} \Delta x_i / \cos \theta & \text{at } \theta \leq \alpha_i, \\ \Delta y_j / \sin \theta & \text{at } \theta > \alpha_i. \end{array} \right. \]  

(38)

The value of the function \(\Phi_b(c_p, \theta)\) is calculated as the linear interpolation of \(\phi\) in the two nearest points of the grid

\[ \Phi_b(c_p, \theta) = \left\{ \begin{array}{ll} (1 - T_{ij}) \phi_{i-1,j} + T_{ij} \phi_{i-1,j-1}, & \text{at } \theta \leq \alpha_i, \\ (1 - T_{ij}^{-1}) \phi_{ij-1} + T_{ij}^{-1} \phi_{i-1,j-1}, & \text{at } \theta > \alpha_i. \end{array} \right. \]  

(39)

where \(T_{ij} = \tan \theta / \tan \alpha_j\). Because of the flow symmetry the calculations can be carried out only in the range \(0 \leq \theta \leq \pi/2\), but
over the whole physical space. The numerical scheme for the function \( \psi \) is the same as that for \( \phi \).

For the first iteration the functions \( \psi, \tau \), \( u \) and \( q \) are assumed to be known. When Eqs.(23) and (24) are solved for all combinations \( c_{pk} \) and \( \theta_m \), then the new quantities of \( \psi, \tau \), \( u \) and \( q \) are calculated according to Eqs.(27)-(31). The quadrature has the following form

\[
\psi(y_i, z_j) = \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \phi_{ij}(c_{pk}, \theta_m) W_k \Delta \theta, \tag{40}
\]

\[
\tau(y_i, z_j) = \frac{2}{3} \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \left[ \phi_{ij}(c_{pk}, \theta_m) (c_{pk}^2 - 1) + \psi_{ij}(c_{pk}, \theta_m) \right] W_k \Delta \theta, \tag{41}
\]

\[
u_i(y_i, z_j) = \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \phi_{ij}(c_{pk}, \theta_m) c_{pk} \cos \theta_m W_k \Delta \theta, \tag{42}
\]

\[
u_z(y_i, z_j) = \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \phi_{ij}(c_{pk}, \theta_m) c_{pk} \sin \theta_m W_k \Delta \theta, \tag{43}
\]

\[
q_y(y_i, z_j) = \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \left[ \phi_{ij}(c_{pk}, \theta_m) (c_{pk}^2 - 2) + \psi_{ij}(c_{pk}, \theta_m) \right] c_{pk} W_k \Delta \theta, \tag{44}
\]

\[
q_z(y_i, z_j) = \sum_{k=1}^{N_c} \sum_{m=1}^{N_\theta} \left[ \phi_{ij}(c_{pk}, \theta_m) (c_{pk}^2 - 2) + \psi_{ij}(c_{pk}, \theta_m) \right] c_{pk} \sin \theta_m W_k \Delta \theta, \tag{45}
\]

where \( c_{pk} \) and \( W_k \) are the Gaussian abscissas and weights, respectively. \( N_c \) and \( N_\theta \) are number of points for the variables \( c_p \) and \( \theta \), respectively.

4 Numerical results

The calculations were carried out for the rarefaction parameter in the range \( 0 \leq \delta \leq 20 \). The maximum bulk velocity over the flow field \( u_{max} \) and its radial position \( r \) are given in Table 1. Its azimuthal position is \( \varphi = 45^\circ \) for all values of the rarefaction parameter \( \delta \). It can be seen that the velocity is zero in the free molecular regime (\( \delta = 0 \)), i.e. the gas is at rest for any temperature distribution. It is also tends to zero in the hydrodynamic limit because the thermal creep, which causes the gas circulation, vanishes at \( \delta \to \infty \). It reaches its maximum value in the slip regime flow, i.e. at \( \delta = 12 \). The radial position of the maximum velocity slightly depends on the rarefaction parameter. The distribution of the bulk speed is shown in Fig.4 together with the streamlines. It can be seen that the speed reaches its largest value near the wall where the surface temperature gradient pushes the gas to the hotter region.

The density distributions are depicted in Fig.5 for some values of the rarefaction parameter. It can be seen that the distribution is antisymmetric with respect to the radius at \( \varphi = \pi/4 \). Its maximum positive deviation is reached at \( r = 1 \) and \( \varphi = 0^\circ \), while the maximum negative deviation is placed at \( r = 1 \) and

\[ 
\]
<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$u_{max}$</th>
<th>$r$</th>
<th>$\rho_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.11E-4</td>
<td>1.</td>
<td>0.126</td>
</tr>
<tr>
<td>0.1</td>
<td>8.87E-4</td>
<td>1.</td>
<td>0.133</td>
</tr>
<tr>
<td>1</td>
<td>3.62E-3</td>
<td>0.96</td>
<td>0.194</td>
</tr>
<tr>
<td>5</td>
<td>9.30E-3</td>
<td>0.94</td>
<td>0.327</td>
</tr>
<tr>
<td>8</td>
<td>1.09E-2</td>
<td>0.94</td>
<td>0.371</td>
</tr>
<tr>
<td>10</td>
<td>1.13E-2</td>
<td>0.94</td>
<td>0.390</td>
</tr>
<tr>
<td>12</td>
<td>1.13E-2</td>
<td>0.94</td>
<td>0.404</td>
</tr>
<tr>
<td>15</td>
<td>1.12E-2</td>
<td>0.94</td>
<td>0.420</td>
</tr>
<tr>
<td>18</td>
<td>1.08E-2</td>
<td>0.94</td>
<td>0.431</td>
</tr>
<tr>
<td>20</td>
<td>1.05E-2</td>
<td>0.94</td>
<td>0.437</td>
</tr>
</tbody>
</table>

$\varphi = 90^\circ$. The value of the maximum deviation $\rho_{max}$ are given in Table 1 as function of the rarefaction parameter $\delta$. The deviation of the density increases by increasing the rarefaction parameter. In the hydrodynamic limit ($\delta \to \infty$) the deviation is related to the temperature deviations as $\rho + \tau = 0.5$, because the pressure is constant over the cross section.

The temperature distributions are depicted in Fig.6, respectively. It can be seen that for the small value of the rarefaction parameter, i.e. $\delta = 0.01$, the temperature slightly differs from its average value $\bar{\tau} = 0.5$. The deviation from the average value increases by increasing the rarefaction parameter.

5 Concluding remarks
The distributions of the bulk velocity, density and temperature inside of a tube due to the temperature variation over its wall was calculated over the rarefaction parameter range from 0.01 to 20. It was found that the bulk velocity reaches its maximum value in the slip regime of the flow, i.e. at $\delta = 20$.

In the future work, some other temperature distribution will be considered.

ACKNOWLEDGMENT
The authors acknowledge the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil) for the support of his research.

REFERENCES
Figure 4. Streamlines and velocity distribution, $u(y,z)$
Figure 5. Density deviation distribution, $\rho(y, z)$. 

Copyright © 2009 by ASME
Figure 6. Temperature deviation distribution, $\tau(y, z)$