Appendix E
Exponential Growth and Doubling Time

This material, adapted from papers written by the late Al Bartlett, makes a fine lecture, for the material is not only very important, but is fascinating—and very wide in scope. It can nicely follow Chapter 15, after discussion of climate change and continued industrial growth. Or it can be coupled to a discussion of radioactive half-life as treated in Chapter 33. Or it can be treated in any break—following an exam, perhaps, or on any day that lends itself to a departure from chapter material. As Al admonished, the greatest shortcoming of the human race is our inability to understand the exponential function. I might add, to understand the implications of the exponential function.

The concept of growth rate can be expressed in simple steps:

Step 1: (new amount) = (old amount) + k times (old amount).
Step 2: (new amount) becomes (old amount).
Step 3: Keep repeating.

That's it. The mathematics is just arithmetic. Use positive k for growth, and negative k for decay.

A beginning application is simple 10% annual interest on each dollar that in previous years was in vogue in a savings account. At the end of the 1st year, A = 1 + 0.10(1); 2nd year, A = 1.10 + 0.10(1.10); 3rd year, A = 1.21 + 0.10(1.21); and so on. Suppose your savings are silver dollars and the bank charges 10% annual storage fee.

	INTEREST		RENTAL	
Year	Change	Amount	Change	Amount
0		1.00		1.00
1	+0.100	1.10	-0.100	0.90
2	+0.110	1.21	-0.090	0.81
3	+0.121	1.33	-0.081	0.73
4	+0.133	1.46	-0.073	0.66
5	+0.146	1.61	-0.066	0.59
6	+0.161	1.77	-0.059	0.53
7	+0.177	1.95	-0.053	0.48
8	+0.195	2.14	-0.048	0.43
9	+0.214	2.36	-0.043	0.39
10	+0.236	2.59	-0.039	0.35
20	+0.612	6.73	-0.014	0.12

Note that in 7 years at a 10% rate the amount just about doubles for positive k and just about halves for negative k.

It is customary to use the decay halving time (half-life) of processes such as radioactive decay as a property of the decaying elements. There is nothing special about doubling-halving time. Tripling-thirding or 3/2ing—2/3ing, or any factor and its reciprocal could be used. As the number of time intervals increases, the process approaches continuity, which leads to the exponential e^{kt}.

The formula for doubling time in the text appears without derivation, which is likely beyond the scope of a nonscience physics class. Its derivation is as follows: Exponential growth may be described by the equation

$$A = A_o e^{kt}\ .$$

where k is the rate of increase of the quantity A_o. We can re-express this for a time T when $A = 2A_o$,

$$2A_o = A_o e^{kT}\ .$$

If we take the natural logarithm of each side we get

$$\ln 2 = kT \quad \text{where } T = \frac{\ln 2}{k} = \frac{0.693}{k}\ .$$

If k is expressed in percent, then

$$T = \frac{69.3}{\%} \sim \frac{70}{\%}\ .$$

When percentage figures are given for things such as interest rates, population growth, or consumption of nonrenewable resources, conversion to doubling time greatly enhances the meaning of these figures.

Next-Time Questions:
• Growing Beanstalk
• Paper Fold

Hewitt-Drew-It Screencast: •*Exponential Growth*

ANSWERS TO APPENDIX E (Exponential Growth and Doubling Time)

1. Half covered on day 29; one-quarter covered on day 28—the coverage doubles each day.

2. A dollar loses half its value in one doubling time of the inflationary economy; this is 70/3.5% = 20 years.

3. At a steady inflation rate of 3.5%, the doubling time is 70/3.5% = 20 years; so every 20 years the prices of these items will double. This means the $50 dollar theatre ticket in 20 years will cost $100, in 40 years will cost $200, in 60 years will cost $400. The $500 coat will similarly jump each 20 years to $1000, $2000, and $4000. For a $50,000 car the 20-year jumps in price give $100,000, $200,000, and $400,000. For a $500,000 home, the 20-year jumps in price are $1,000,000, $2,000,000, and $4,000,000!

4. For a 5% growth rate, 42 years is three doubling times (70/5% = 14 years; 42/14 = 3). Three doubling times is an eightfold increase. So in 42 years the city would have to have 8 similar sewage treatment plants to service 8 times the population.

5. All things being equal, doubling of food for twice the number of people simply means that twice as many people will be eating, and twice as many will be starving as are starving now.

6. Doubling one penny for 30 days yields a total of $10,737,418.23.

7. On the 30th day your wages will be $5,368,709.12, which is one penny more than the $5,368,709.11 total from all the preceding days.

8. It is generally acknowledged that if the human race is to survive, even from an overheating of the world standpoint, while alleviating even part of the misery that afflicts so much of humankind, the present rates of energy consumption and population growth must be reduced. The chances of achieving reduced growth rates are better in a climate of scarce energy than in a climate of abundant energy. We must hope that by the time we have fusion under control, we will have learned to optimize our numbers and to use energy more wisely.