System-environment correlations for dephasing two-qubit states coupled to thermal baths

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I. INTRODUCTION

Quantum coherences and multipartite quantum correlations are essential resources for quantum information processing [1]. However, realistic carries of quantum information are never isolated and have to be treated as open quantum systems [2,3]. The possibility to control and manipulate quantum information is limited by decoherence and dissipation, usually caused by the environment coupled to the central quantum system. Thus, a thorough understanding of the dynamics of entanglement in open systems [4–6] is of fundamental importance. Most investigations focus on the dynamics of the reduced quantum system, tracing over the environmental degrees of freedom. Obviously, the build-up of system-environment (SE) correlations cannot be discussed on the reduced level.

Decoherence and loss of entanglement in an open quantum system is due to the build-up of SE correlations. To properly study such correlations, the environmental degrees of freedom have to be taken into account, based on a model for system, environment, and their interactions [7–11]. Some work has been done in the context of total models for entangled qubit systems and the appearance of correlations with the surrounding environments [12,13]. There, the authors find that complete loss (sudden death) of entanglement in the system can manifest itself before, simultaneously, or even after the sudden birth of entanglement with the environment. More recently, entanglement in reduced bipartitions has been studied for systems coupled with pure environment initial states [14–16].

It seems natural to assume that the loss of entanglement in an open system is accompanied by the build-up of entanglement with the environment. One may think of a transfer of entanglement from the local to the total state. This is certainly true for a pure total state. However, for a finite temperature bath it is less obvious. For realistic finite temperature baths it is a challenge to prove entanglement or separability for the total state.

It is an interesting question whether the SE correlations are of quantum or of classical nature. In general it is hard to answer this question for realistic environments. Therefore, to fully understand the decoherence process, and to be able to control the above mentioned resources, it is desirable to describe the detailed dynamics of SE quantum correlations. One possibility is to extract, for example, information about the SE correlations using monogamic relations for the main system [17,18]. A more complete description of the full dynamics, however, is obtained from the total state (system plus environment) by using a coherent basis and a partial representation of the total density operator [19,20].

In addition, it is worth mentioning that SE quantum correlations depend on a physically appropriate dilation of qubit dephasing, where for different environments we obtain the same dephasing dynamics for the reduced system. In other words, different total systems and SE correlations lead to the same reduced dynamics. Therefore, in order to properly investigate SE correlations, it is important to choose the appropriate dilation for the total system.

Here we are interested in a detailed study of how SE correlations build up when decoherence and loss of entanglement occur in the central system. For this we analyze the model of two qubits, coupling one of them to an environment of dephasing nature. This is an extension of previous investigations [8,9] where only one qubit was coupled to thermal baths composed by harmonic oscillators. In fact, since it is an analytically solvable model, we are able to construct conditions for SE separability and entanglement, and to compare them with the entanglement present in the two-qubit system.

This work is organized as follows. In Sec. II we discuss the dilation of qubit dephasing for different environments. Sections III and IV present our model and give an exact expression for the total state of the system. Using this expression we construct criteria for separability and entanglement between the system and the environment. While some results
for distinct initial Werner states and coupling strengths are presented in Sec. V, Sec. VI discusses entanglement within many different bipartitions. We reserve Sec. VII for discussion and conclusions.

II. DILATING QUBIT DEPHASING

In order to study the build-up of SE correlations, we have to specify a physical realization of the environment and the SE interaction. More precisely, we here are not only interested in general correlations but in SE entanglement. This section serves as a simple introduction to clarify the relevance of the choice of dilation when studying SE correlations, in particular for mixed environmental initial states. In any case, we need to determine the total SE state.

Locally, within the framework of completely positive (CP) and trace preserving maps, single-qubit dephasing in the computational basis is given by the quantum channel \[1\],

\[
\rho \rightarrow \rho' = \mathcal{E}[\rho] = \frac{1 + \sqrt{p}}{2} \rho + \frac{1 - \sqrt{p}}{2} (\sigma_3 \rho \sigma_3), \tag{1}
\]

with the third Pauli matrix \(\sigma_3\), or, in matrix notation,

\[
\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \rightarrow \rho' = \begin{pmatrix} \rho_{00} & \sqrt{p} \rho_{01} \\ \sqrt{p} \rho_{10} & \rho_{11} \end{pmatrix}. \tag{2}
\]

Here, the real \(p\) with \(0 \leq p \leq 1\) takes the role of the dephasing parameter: \(p = 1\) corresponds to no dephasing, while \(p = 0\) describes the full loss of coherence. Often, \(p\) will be some (decaying) function of time, depending on coupling strength and, for a thermal bath, on the temperature of the environment (see later).

Before discussing various dilations of the dephasing channel (1), it is worth noting that all unital single-qubit channels (including dephasing) are of so-called random-unitary (RU) type, i.e., they can be obtained from an ensemble of unitary evolutions without invoking a quantum environment at all [21]. It is only for two-qubit systems (and larger) that dephasing may be of true quantum nature [22,23].

Thus, the local point of view (1) does not allow for any conclusions about SE correlations. We need to specify the underlying total dynamics \((U_{\text{tot}})\) and the environmental initial state \(\rho_{E}\). Then, the dilation,

\[
\rho' = \mathcal{D}[\rho] = \text{Tr}_{E}[U_{\text{tot}}(\rho \otimes \rho_{E}) U_{\text{tot}}^\dagger]], \tag{3}
\]

allows us to study SE correlations. Clearly, depending on the choice of dilation, different SE correlation scenarios are possible, as we will point out next.

For pure qubit dephasing studied here, without loss of generality the total unitary evolution can be written in the form [24],

\[
U_{\text{tot}} = |0\rangle \langle 0| \otimes 1_E + |1\rangle \langle 1| \otimes U_E, \tag{4}
\]

with the two “open system” qubit states \(|0\rangle\) and \(|1\rangle\) and a unitary evolution operator \(U_{E}\) of the environment, conditioned on the qubit state \(|1\rangle\). Thus, any dilation of qubit dephasing is fully determined by the initial state \(\rho_{E}\) and a unitary evolution operator \(U_{E}\) of the environment. As we will see, for the build-up of SE entanglement, the purity of the environmental initial state is of great relevance.

A. Pure environmental initial state: entangling dilation

Often, a pure environmental state \(\rho_{E} = |0_E\rangle \langle 0_E|\) is assumed [14–16,24]. Then, by construction, the total state dynamics is entirely determined from the two equations,

\[
U_{\text{tot}}(0)|0_E\rangle = |0\rangle |0_E\rangle, \quad U_{\text{tot}}(|1\rangle |0_E\rangle) = |1\rangle (\sqrt{p} |0_E\rangle + \sqrt{1 - p} |1_E\rangle). \tag{5}
\]

where the relation,

\[
U_{E}|0_E\rangle = \sqrt{p} |0_E\rangle + \sqrt{1 - p} |1_E\rangle, \tag{6}
\]

defines \(p\) and the environmental state \(|1_E\rangle\) on the right-hand side of the equation (we neglect a possible, yet irrelevant phase here). Thus, for a pure environmental initial state \(|0_E\rangle\), only one orthogonal environmental state \(|1_E\rangle\) [as defined in (6)] is relevant and the true environment can be effectively described by a single qubit, as in [14–16]. This single-qubit environment dilation is thus defined by the two choices,

\[
\rho_{E} = |0_E\rangle \langle 0_E|, \quad U_{E} = \sqrt{p} \sigma_3 + \sqrt{1 - p} \sigma_1. \tag{7}
\]

It is easy to show that the partial transpose of the corresponding effective two-qubit SE state has a determinant of

\[
\text{det} \left\{ \rho_{\text{tot}}^{\text{PT}} \right\} = -(1 - p)^2 \rho_{00} \rho_{11} |\rho_{01}|^2. \tag{8}
\]

Using the Peres criterion [25] we conclude that starting from a pure environmental initial state \(|0_E\rangle\), the effective SE two-qubit state will be entangled for both, \(p < 1\) and the initial \(\rho_{01} \neq 0\), i.e., whenever some initial coherence is present and dephasing actually happens.

B. Mixed environmental initial state: separable dilation

Such SE entanglement need not develop for a mixed environmental initial state [8,9]. To give a simple example, consider again a single-qubit dilation of the dephasing channel (1), now with

\[
\rho_{E} = \frac{1 + \sqrt{p}}{2} |0_E\rangle \langle 0_E| + \frac{1 - \sqrt{p}}{2} |1_E\rangle \langle 1_E|, \quad U_{E} = \sigma_3. \tag{9}
\]

In contrast to the pure dilation (7), where the parameter \(p\) may be interpreted as representing time, here \(p\) should be rather interpreted as a measure for initial environmental temperature (large \(p \rightarrow 1\) corresponding to low \(T \rightarrow 0\) and vice versa) and dynamics evolves for a fixed time. As with (7), it is easily checked that (9) is a valid dilation of pure dephasing, leading to the CP map (1) for the reduced state of the qubit. Now, using (4), and in contrast to the previous dilation based on (7), the total state is separable for all \(p\),

\[
\rho_{\text{tot}} = \frac{1 + \sqrt{p}}{2} \rho \otimes |0_E\rangle \langle 0_E| + \frac{1 - \sqrt{p}}{2} (\sigma_3 \rho \sigma_3) \otimes |1_E\rangle \langle 1_E|, \tag{10}
\]

i.e., no SE entanglement builds up.

We conclude from these considerations that in order to study SE entanglement, it is crucial to choose a physically appropriate dilation. Matters are considerably more involved for a mixed environmental initial state: Even for dephasing
we can no longer expect to describe the environment by an effective qubit. Indeed, the action of $U_{\text{tot}}$ [or rather $U_{\text{E}}$ in (4)] is no longer restricted to a single initial environmental state $|0_{\text{B}}\rangle$. Thus, in general, the dynamically relevant environmental Hilbert space can no longer be spanned by just two states as in (6).

In the following, we study a two-qubit system, coupled to a proper environment consisting of an infinite number of degrees of freedom and initially in a thermal (mixed) state. As the total state is neither a two-qubit nor a Gaussian state, the detection of entanglement is a nontrivial issue and will be based on the negativity of the partial transpose [25].

III. MODEL FOR SYSTEM AND ENVIRONMENT

We study an initially entangled two-qubit (qubits $A$ and $B$) state. One of the qubits ($A$) is coupled to its (local) environment—representing a single-qubit dephasing channel for qubit $A$. In Fig. 1, we can see a scheme of the underlying model. The environment is chosen to be a bath of harmonic oscillators, initially in a thermal state, a standard model for open quantum system dynamics [26,27]. As we aim at pure dephasing, the Hamiltonian of the system and the system part diagonal in the chosen (computational) basis. Thus, for the open quantum system dynamics [26,27]. As we aim at pure dephasing, the Hamiltonian of the system and the system part diagonal in the chosen (computational) basis. Thus, for the total Hamiltonian we write $H_{\text{tot}} = H_{\text{sys}} + H_{\text{int}} + H_{\text{env}}$, with a system Hamiltonian,

$$H_{\text{sys}} = \frac{\hbar \Omega_A}{2} (\sigma_3^A \otimes 1^B) + \frac{\hbar \Omega_B}{2} (1^A \otimes \sigma_3^B),$$

(11)
a (bosonic) bath of harmonic oscillators (labeled by $\lambda$ and creation and annihilation operators $a_{\lambda}, a_{\lambda}^{\dagger}$ with commutation relations $[a_{\lambda}, a_{\mu}^{\dagger}] = \delta_{\lambda \mu}$). $H_{\text{env}} = \sum_\lambda \hbar \omega_\lambda a_{\lambda}^{\dagger} a_{\lambda}$, and diagonal (with respect to the system) interaction between qubit $A$ and the environment of the form,

$$H_{\text{int}} = \left( \sigma_3^A \otimes 1^B \right) \otimes \sum_\lambda \hbar (g_\lambda^A a_{\lambda}^{\dagger} + g_\lambda a_{\lambda}).$$

(12)

$\Omega_{A,B}$ are the characteristic frequencies of the qubits and the coefficients $g_\lambda$ are the coupling amplitudes between the qubit $A$ and each environmental mode of frequency $\omega_\lambda$. The full SE Hamiltonian underlying our investigations represents a standard model of quantum dephasing of qubits and can be found in numerous earlier works—we, for instance, [28–30].

We assume that the environment is initially in a thermal state at temperature $T$, expressed by the canonical density operator,

$$\rho_{\text{therm}} = \frac{1}{Z} \exp(-H_{\text{env}} / k_B T),$$

(13)

with partition function $Z = \text{Tr}[\exp(-H_{\text{env}} / k_B T)]$. The mean thermal occupation number is the usual $\bar{n}_\lambda = (\exp[\hbar \omega_\lambda / k_B T] - 1)^{-1}$. The total initial state is the product,

$$\rho_{\text{init}}(0) = \rho_{\text{sys}}(0) \otimes \rho_{\text{therm}},$$

(14)

with a (possibly entangled) initial two-qubit state $\rho_{\text{sys}}(0)$.

In fact, in what follows, due to their analytical tractability, we choose the so-called X states [31]. Thus, the two-qubit initial state in the computational basis is given by the matrix,

$$\rho_{\text{sys}}(0) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix},$$

(15)

where $\sum_{i=1}^{4} \rho_{ii} = 1$. The family of $X$ states includes pure Bell states and the well-known Werner states [32]. Crucially, the family of $X$ states is closed under dephasing dynamics.

The chosen model allows us to derive an exact master equation [8] for the reduced density operator $\rho_{\text{sys}}(t) = \text{Tr}_{\text{env}}[\rho_{\text{tot}}(t)]$ which reads

$$\rho_{\text{red}} = -i \Omega_A \left[ \sigma_3^A \otimes 1^B, \rho_{\text{red}} \right] - i \Omega_B \left[ 1^A \otimes \sigma_3^B, \rho_{\text{red}} \right] - \frac{\gamma_{\text{dph}}(t)}{2} \left( \rho_{\text{red}} - \left( \sigma_3^A \otimes 1^B \right) \rho_{\text{red}} \left( \sigma_3^A \otimes 1^B \right) \right).$$

(16)

Equation (16) takes the form of a master equation of Lindblad type with, however, a time-dependent dephasing rate $\gamma_{\text{dph}}(t) \equiv \gamma$. Indeed, in terms of the environmental spectral density $J(\omega) = \sum_\lambda |g_\lambda|^2 \delta(\omega - \omega_\lambda)$, the dephasing rate is given by

$$\gamma_{\text{dph}}(t) = 4 \int_0^t ds \int_0^\infty d\omega J(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \cos(\omega s).$$

(17)

Note that this rate may turn negative at times for nontrivial spectral densities and therefore the CP map $\rho(0) \rightarrow \rho(t)$ may lose its divisibility, which is used as an indication for non-Markovian quantum dynamics [9,33,34].

The solution of Eq. (16) with initial state (15) is the reduced state,

$$\rho_{\text{red}}(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \rho_{14}^* \rho_{22} & \rho_{23} \rho_{23}^* & 0 & \rho_{23} \rho_{44}^* \\ 0 & \rho_{22}^* \rho_{23}^* & \rho_{33} & 0 \\ 0 & \rho_{23}^* \rho_{44}^* & \rho_{44} & 0 \\ \rho_{14} \rho_{14}^* \rho_{23} \rho_{23}^* & 0 & \rho_{44} & 0 \end{pmatrix},$$

(18)

where $\rho_{\text{sys}}(t) = \exp \left[ i(\Omega_A \pm \Omega_B) - \int_0^t \gamma_{\text{dph}}(s) ds \right]$.

Entanglement within the two-qubit system can be calculated via concurrence [35], a well-known measure for mixed two-qubit states. For our model this measure of entanglement is given by

$$\mathcal{C}(\rho_{\text{red}}(t)) = 2 \max \{ 0, |\rho_{23}| \rho_{23}^* (t) \} \left( 1 - \sqrt{\rho_{11}^2 \rho_{44}^2 + \rho_{14}^2 \rho_{23}^2} \right).$$

(19)
where $|\mathcal{S}(t)| = |\mathcal{S}_0(t)|$. Concurrence varies between 0 (separable states) and 1 (maximally entangled states). Clearly, due to the dephasing dynamics, $\mathcal{C}$ typically decreases with time. Moreover, as is apparent from (19), entanglement may disappear entirely, even for a finite time (finite $|\mathcal{S}(t)|$)—sometimes referred to as sudden death [31].

On the reduced level, the loss of entanglement in a two-qubit state due to a local dephasing channel has been studied in many publications [31,36,37]. That loss is accompanied by the build-up of correlations between the open quantum system and its environment. In order to study whether initial entanglement within the open two-qubit state just disappears or whether it is transferred to entanglement between system and environment (or to entanglement within the elements of the environment), we need to determine the total state.

IV. TOTAL STATE DYNAMICS

The study of correlations between the system and the infinite oscillator environment requires an expression for the total state. For this purpose we use a coherent state basis for the environmental degrees of freedom and choose a partial representation [20]. We here follow closely a similar analysis for the dephasing of a single qubit presented in [8,9]. The total state is written as

$$\rho_{tot}(t) = \int \frac{d^2 z}{\pi} \frac{1}{\hat{n}} e^{-|z|^2/\hat{n}} \hat{P}(t; z, z^*) \otimes |z\rangle \langle z|,$$

where $z = (z_1, z_2, \ldots)$ is a vector of complex numbers representing environmental coherent state labels and we use the notation $d^2 z/\pi = d^2 z_1/\pi d^2 z_2/\pi \ldots$ and $|\alpha| = \prod_n \exp(-|z|^2/\hat{n}) = \prod_n \exp(-|z|^2/\hat{n})/\hat{n}_\lambda$. Note that for $t = 0$, we have the initial $\hat{P}(t=0) = \rho_{sys}(0) = \rho_{mcd}(0)$ such that (20) represents the initial state (14). It is only for $t > 0$ that $\hat{P}(t)$ becomes $z$-dependent and, thus, expression (20) represents a correlated SE state.

Expression (20) is a solution of the total von Neumann equation. We find the time evolution of the partial P function to be given by

$$\hat{P}(t; z, z^*) = \begin{pmatrix} \mathcal{A} + \rho_{11} & 0 & 0 & \mathcal{B} + \rho_{14} \\ \mathcal{A} + \rho_{21} & 0 & 0 & \mathcal{B} + \rho_{24} \\ \mathcal{A} + \rho_{31} & 0 & 0 & \mathcal{B} + \rho_{34} \\ \mathcal{A} + \rho_{41} & 0 & 0 & \mathcal{B} + \rho_{44} \end{pmatrix},$$

where $\mathcal{A}(t) = \exp[-i(\lambda t + \mathcal{A} + \rho_{44})]$, $\mathcal{B}(t) = e^{i\Omega A t} e^{i\Omega B t} \exp[i\beta(t)] - (b(t)|z\rangle - |b(t)|z\rangle)$. Here $a(t) = (a_1(t), a_2(t), \ldots)$ and $b(t)$ are complex time-dependent vectors,

$$a_\lambda(t) = \frac{1}{\hat{n}_\lambda} \int_0^t (g_\lambda e^{i\omega t}) ds,$$

$$b_\lambda(t) = \int_0^t (g_\lambda e^{i\omega t}) ds,$$

with the scalar product $\langle a(t)|z\rangle = \sum_\lambda a_\lambda(t)\delta_{\lambda 0} z_\lambda$ and

$$\alpha(t) = 2\mathrm{Re} \int_0^t ds \int_0^t d\tau \left[ \sum_\lambda \frac{1}{\hat{n}_\lambda} |g_\lambda|^2 e^{-i\omega(t-\tau)} \right].$$

A. System-environment separability

Expression (20) allows us to study correlations between system and environment. In particular, it is clear that it is a separable representation of the total state whenever $\hat{P}$ is a positive two-qubit matrix [8].

Separability criterion. As long as the partial $P$ function is positive semidefinite, the total state $\rho_{tot}(t)$ in representation (20) is trivially separable. Being of X type, the eigenvalues can be determined analytically. Initially, all eigenvalues are positive and they remain positive as long as

$$\mathcal{S}(t) \leq \frac{1}{2} \ln \left( \frac{\rho_{22}\rho_{33}}{|\rho_{23}|^2} \right) \quad \text{and} \quad \mathcal{S}(t) \leq \frac{1}{2} \ln \left( \frac{\rho_{11}\rho_{44}}{|\rho_{14}|^2} \right),$$

where we have defined

$$\mathcal{S}(T,t) := \alpha(t) + \beta(t)$$

$$= 4 \int_0^t ds \int_0^t d\tau \int_0^\infty d\omega \times J(\omega)e^{-\omega s} \cos[\omega(s - \tau)].$$

Depending on the choice of the spectral density $J(\omega)$, Eq. (23) can be written in terms of known special functions. Thus, as long as (22) is satisfied (as a function of time and temperature), the total state is separable and no SE entanglement builds up, even though the initial two-qubit state may well lose its initial entanglement (see results later).

B. System-environment entanglement

The detection of entanglement between system and environment is very demanding since the underlying state is infinite dimensional and not of Gaussian type. We can use the Peres criterion [25] to see that entanglement is there, surely, if the partial transpose $\rho_{tot}(t)$ of the total state has a negative expectation value $\mathcal{E}^{PT} = \langle \Psi| \rho_{tot}^{PT} |\Psi\rangle$ for some suitably chosen total system state $|\Psi\rangle$.

We expand $|\Psi\rangle \propto \int \frac{d^2 z}{\pi} e^{-|z|^2} |\psi(z)\rangle |z\rangle$ in a Bargmann coherent state basis. Thus, in order to detect entanglement, we need to find a system state $|\psi(z^*)\rangle \sim |z\rangle|\Psi(t)\rangle$ analytical in $z^*$ in the Hilbert space of the two qubits such that

$$\mathcal{E}^{PT} \sim \int \frac{d^2 z}{\pi} e^{-\frac{|z|^2}{2}} \langle \psi(z)| \hat{P}^T(z, z^*) |\psi(z^*)\rangle < 0.$$

After some experimentation we choose

$$|\psi(t, z^*)\rangle = \begin{pmatrix} r e^{-i|z|^2/2 + i(\Omega_A + \Omega_B)/2} \\
-s e^{-i|z|^2/2 + i\Omega_A} \\
-t e^{-i|z|^2/2 - i(\Omega_A + \Omega_B)/2} \\
-u e^{-i|z|^2/2 - i\Omega_A} \end{pmatrix}. $$

Here we found that for optimal entanglement detection, the vector $(r, s, t, u)$ needs to be determined as the pure state which has the smallest overlap with the transpose of the initial state of the two-qubit system.
Performing the integral over the coherent state labels $z_i$, we find
\[
ed^{\Pi(t)} = \frac{1}{\bar{n} + 1} \left[ e^{-\omega(t)} e^{\omega(t)} (\rho_{11}|r|^2 + \rho_{22}|s|^2 + \rho_{33}|t|^2 + \rho_{44}|u|^2) + e^{\omega(t)} e^{-\omega(t)} (\rho_{23}s^*t + \rho_{32}s^*t + \rho_{14}|r|^2 + \rho_{14}|r|^2u) \right],
\]
where $\tilde{\mathcal{J}}(t)$ is defined similar to $\mathcal{J}(t)$ in (22), but with $\exp(\hbar\omega/kT)$ replaced by its inverse, $\exp(-\hbar\omega/kT)$.

From (26) we conclude that SE entanglement is surely present whenever the following condition is satisfied:
\[
\mathcal{E}(T,t) = \mathcal{J}(T,t) - \tilde{\mathcal{J}}(T,t) > \ln \left[ \frac{-\rho_{11}|r|^2 + \rho_{22}|s|^2 + \rho_{33}|t|^2 + \rho_{44}|u|^2}{\rho_{23}s^*t + \rho_{32}s^*t + \rho_{14}|r|^2 + \rho_{14}|r|^2u} \right].
\]

The relevant quantity on the left-hand side of (27) can be written in terms of the spectral density as
\[
\mathcal{E}(T,t) = 8 \int_0^T ds \int_0^s dt \int_0^\infty d\omega \times J(\omega) \sin(h\omega/k_BT) \cos[\omega(s - t)].
\]

Thus, with (27) we found a criterion that allows us to detect SE entanglement as a function of time and temperature of the bath.

In the next section we will show the regions of SE separability and entanglement in the temperature-time diagram $(T,t)$, defined by conditions (22) for separability, and (27) for entanglement. Let us already mention at this point that these two conditions do not fill the whole $(T,t)$ plane. There will be $(T,t)$ combinations where we cannot make any statement about whether the total state is entangled or separable. This is due to the fact that our criteria are sufficient, but not necessary conditions. For entanglement on one hand, even if our test state (25) were optimal, there may well be entangled states with positive partial transpose (“bound entangled states” [38]). For separability, on the other hand, there may well be separable states with a negative partial $\bar{\mathcal{P}}$ representation.

V. NUMERICAL RESULTS FOR WERNER STATES

In this section we present and discuss concrete results concerning entanglement and separability of the total SE state. Choosing an Ohmic spectral density with a cutoff frequency $\omega_c$, with $J(\omega) = k\omega\Theta(\omega - \omega_c)$, where $\kappa$ is the coupling strength between system and environment, we can construct diagrams of separability and entanglement for different initial states and coupling strengths (varying $\kappa$).

Furthermore, we want to compare the time for the decay of initial entanglement and the time for the build-up of SE entanglement with the typical decoherence time of the open two-qubit system. The latter time, $T_{\text{dec}}$, we define through the relation $\int_{-\infty}^{\infty} \gamma_{\text{dec}} ds = 1$, which defines the time scale of the decay of the off-diagonal elements of the reduced operator in Eq. (18).

As initial two-qubit $X$ states we choose Werner states. These are a family of states that depend on a single purity parameter $c$, and are given by $\rho^W = \frac{1}{4} I^{AB} + c \left| \phi^- \right\rangle \left\langle \phi^- \right|$, where $I^{AB}$ is the identity matrix in the Hilbert space of the two qubits and the Bell state $\left| \phi^- \right\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, with $c \in [0,1]$. This state is entangled if $c > 1/3$.

For this specific state the condition for separability, from Eq. (22), is
\[
\mathcal{E}(T,t) = \ln \left[ \frac{1 + c}{2c} \right],
\]
and for entanglement, from Eq. (27), is
\[
\mathcal{E}(T,t) > \ln \left[ \frac{1 + c}{2c} \right].
\]

The pure state that has the smallest overlap with the transpose of the two-qubit system initial state is
\[
(r,s,t,u) = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right).
\]

In the following we discuss weak SE interaction first ($\kappa = 10^{-3}$), followed by strong coupling ($\kappa = 1$). In both cases we choose three different initial two-qubit states: $c = 0.2$ (no initial two-qubit entanglement, low purity), $c = 0.5$ (some initial entanglement, medium purity), and $c = 0.9$ (large initial entanglement, fairly pure). In all cases we see an initial phase where the total state remains separable. For very low temperatures this phase is hardly visible and SE entanglement builds up quickly. We also display the loss of initial two-qubit entanglement and initial two-qubit coherence that is observed in all cases. Numerical results are presented in temperature-time $(T,t)$ diagrams following the notation: red (black gray) color when the entanglement condition [Eq. (29)] is satisfied and blue (light gray) color when the separability condition [Eq. (29)] is satisfied.

A. Weak coupling ($\kappa = 10^{-3}$)

We start with an initial Werner state with parameter $c = 0.2$. There is never any entanglement between the two qubits, but we observe from Fig. 2 that for low temperatures, due to SE interaction, entanglement between the two-qubit system and environment builds up [red (dark gray) region]. There is a clearly visible boundary at a critical temperature $k_BT_{\text{crit}} \approx 0.13h\omega_c$, above which the total SE state remains separable for all times displayed in the figure [blue (light gray) region].

Interestingly, in the vicinity of the critical temperature, we see oscillations as a function of time between entangled and separable regions. To better understand the appearance of the critical temperature and the oscillations, let us note that for our choice of spectral density we can perform the integrals in Eq. (27). In this way we obtain
\[
\mathcal{E}(T,t) = 8\kappa \text{Shi} \left[ \frac{h\omega_c}{k_BT} \right] - 4\kappa t \sinh \left[ \omega_c \left( t + \frac{i\hbar}{k_BT} \right) \right]
\]
\[
- \text{Shi} \left[ \omega_c \left( t + \frac{i\hbar}{k_BT} \right) \right]
\]
\[
\approx 8\kappa \text{Shi} \left[ \frac{h\omega_c}{k_BT} \right] - \frac{2\kappa(\omega_c t)}{(\omega_c t)^2 + \left( \frac{h\omega_c}{k_BT} \right)^2} e^{\frac{h\omega_c}{k_BT}} \sin(\omega_c t),
\]
\[
(\text{as } t \to \infty).
\]
FIG. 2. Temperature-time diagram. Red (dark gray) region, entangled SE state; blue (light gray) region, separable SE state. Parameters are $\kappa = 10^{-3}$ (weak coupling) and $c = 0.2$ (no initial entanglement between the two qubits). Remarkably, for $k_B T_{\text{crit}} \approx 0.13 \hbar \omega_c$, the total state oscillates as a function of time between separable and entangled regions.

Looking at the entanglement criterion for Werner states (30), the critical temperature is determined from

$$8\kappa \text{Shi} \left[ \frac{\hbar \omega_c}{k_B T_{\text{crit}}} \right] = \ln \left[ \frac{1 + c}{2c} \right],$$

(33)

which gives a numerical value of $k_B T_{\text{crit}} \approx 0.1345 \hbar \omega_c$ (see Fig. 2). This expression shows the dependence of the critical temperature on the initial two-qubit state.

In Fig. 3 we analyze the case when the two qubits are initially entangled [$c = 0.5$ in Fig. 3(a) and $c = 0.9$ in Fig. 3(b)]. Again, we see SE entanglement building up for very low temperature, and observe a separable total state for larger temperatures and all times displayed in the figures. The black full line indicates, for a given temperature, the time of complete loss of two-qubit entanglement (sudden death). Accordingly, the black dashed line indicates the decoherence time scale of the two-qubit state. The critical temperatures for SE entanglement are (a) $k_B T_{\text{crit}} \approx 0.16 \hbar \omega_c$ and (b) $k_B T_{\text{crit}} \approx 0.29 \hbar \omega_c$, which means that the total SE state remains separable (besides the small oscillations) for all displayed times and $T > T_{\text{crit}}$.

Interestingly, our results show that the loss of entanglement between the two qubits does not have a direct relation with the build-up of SE entanglement. Unless the temperature is extremely low, we see that the decay of entanglement (and also decoherence) in the system happens while the SE bipartition is still in a separable state. While for the highly entangled, rather pure state ($c = 0.9$) the decoherence time scale is shorter than the time for sudden death, this situation reverses for a less entangled initial state ($c = 0.5$).

B. Strong coupling ($\kappa = 1$)

Here we choose the same three Werner initial states as before. The most significant difference to the weak coupling case is that our criteria for separability and entanglement no longer cover the whole temperature-time diagram. We observe the appearance of a white region, where the separability and entanglement conditions are not sufficient to decide whether the system is entangled with the environment or whether the total state is still separable.

For the initial Werner state with parameter $c = 0.2$ (no entanglement between the qubits), we can see in Fig. 4 that our criteria for entanglement and separability have the same border line for short times ($\omega_c t \lesssim 0.5$). For times $\omega_c t \gtrsim 0.5$ a gap between the separability and entanglement conditions appear, as mentioned above. Even without any entanglement between the qubits, due to SE interaction and the coherences of
the coupled qubit, entanglement between the two-qubit system and environment builds up.

In a more interesting scenario, in Fig. 5 we show the initial Werner states with parameters (a) \( c = 0.5 \) and (b) \( c = 0.9 \). In these cases we have initial entanglement between the qubits, until entanglement sudden death (black full line). As in the case of weak coupling, looking at the figures it is clear that the build-up of SE entanglement does not have any relation with the sudden death of entanglement within the two-qubit states.

For low temperatures, for example, there is still entanglement between the qubits, while the SE is still separable and then entangled. For larger temperatures we can see entanglement sudden death, when SE is still separable, and becomes entangled after some time.

In the context of strong coupling we are also able to calculate the critical temperatures from Eq. (32). These temperatures are \( k_B T_{\text{crit}} \approx 7.29 \hbar \omega_c \) for \( c = 0.2 \), \( k_B T_{\text{crit}} \approx 19.7 \hbar \omega_c \) for \( c = 0.5 \), and \( k_B T_{\text{crit}} \approx 148 \hbar \omega_c \) for \( c = 0.9 \), and lie outside the temperature ranges displayed in the figures.

### C. Decay of concurrence

We point out that oscillations in time between entangled and separable total state, as observed in Fig. 2 for \( T \approx T_{\text{crit}} \), are reflected to some extent in an oscillatory decay of concurrence in the two-qubit state. However, care has to be taken since the same type of oscillations in the concurrence decay are visible in regions of SE entanglement, or as SE separability, respectively.

In Fig. 6 an example of this oscillatory behavior is shown. For an initial Werner state with \( c = 0.5 \) and weak coupling, we see this oscillatory decay of concurrence (19) due to the coupling of qubit \( A \) with the environment. While the dashed curve corresponds to oscillations between entangled and separable total state, the full and dotted curves correspond to entangled and separable SE states, respectively. This behavior
is more accentuated for small temperatures, and becomes smooth for larger temperatures.

VI. OTHER BIPARTITIONS

So far we focused on SE entanglement and its relation to the loss of entanglement within the initial system (\(\rho_{AB}\)) state. Other authors have studied entanglement in reduced bipartitions [14–16] for a pure environmental initial state and found no bipartite entanglement in any of the states \(\rho_{AE} = \text{tr}_B(\rho_{ABE})\) or \(\rho_{BE} = \text{tr}_A(\rho_{ABE})\). These findings also hold in our case, where a thermal environmental initial state is used.

Tracing over the spectator qubit \(B\), \(\rho_{AE} = \text{tr}_B(\rho_{ABE})\), there is never any entanglement between \(A\) and the \(E\) environment. This is because we have chosen \(X\) states whose reduced \(A\) state is a diagonal mixture in the dephasing basis and initial separability is preserved.

Tracing over qubit \(A\), the state \(\rho_{BE} = \text{tr}_A(\rho_{ABE})\) will never develop any entanglement between \(B\) and the \(E\) environment. Again this is due to the fact that there is no initial entanglement and \(B\) is the spectator qubit.

In [15,16] the authors investigate tangle for the three-qubit \(ABE\) state, showing GHZ-type three-qubit entanglement. Let us stress, however, that for a mixed environmental initial state the three-qubit picture ceases to hold and statements about multipartite entanglement are difficult to obtain. Nevertheless, we see in Fig. 5 that for long enough times there is entanglement between the two qubits and environment, while all three bipartite states \(\rho_{AB}, \rho_{AE}, \rho_{BE}\) are separable, pointing at genuine three partite entanglement.

In [14], the authors couple a system of two qubits (\(X\) state initially entangled) to two independent environments \(\rho_{ABE_1E_2}\), which they assume to be qubits. By looking at the dynamics of correlations for different bipartitions, they show that there does not exist any relation between the build-up of SE correlation and the decay of entanglement in the main system. However, they do not study the partition \(AB - E\), where we find the build-up of SE (quantum) correlations.

In the present work and others discussed above [14–16], the main message is that apparently there is no simple relation between the loss of entanglement within the system and the build-up of entanglement with the environment. We cannot identify a transfer of entanglement from the system to the SE partition.

Can the lost entanglement of the system be found within the environment? The answer is no, as can be easily checked in our model.

Tracing over the qubits we find the \(P\) representation of the environmental state,

\[\rho_E = \text{Tr}_{XY}(\rho_{tot}(t)) = \int \frac{d^2z}{\pi} \frac{1}{n} e^{-i\xi^2/\hbar} \hat{P}_E(t;\xi,z^*)|z\rangle\langle z|,\]

with the positive,

\[\hat{P}_E(t;\xi,z^*) = |\alpha_+^E(\rho_{11} + \rho_{22}) + \alpha_-^E(\rho_{33} + \rho_{44})| > 0.\]

From the positivity of \(\hat{P}_E(t;\xi,z^*)\) we can conclude that any multimode reduced state of the environment is a classical mixture of coherent states and thus separable. In particular, any bipartite state of any two modes \(\lambda_1, \lambda_2\) picked from the environment, is separable. To conclude, there is never any build-up of entanglement between the modes of the environment, indicating that indeed the entanglement between the qubits may completely disappear.

VII. CONCLUSIONS

The understanding of system-environment correlations is fundamental for preserving quantum information. In this work we investigate the dynamics of quantum correlations (entanglement) and study how they are redistributed (or not) among distinct partitions.

Any investigation of system-environment correlation starts with the choice of the full system-environment model. Therefore, we first discuss the issue of dilating qubit dephasing, highlighting physically different models which lead to the same dephasing dynamics on the reduced level.

Here, we use a realistic, finite temperature, infinite degrees of freedom environment. We investigate the entanglement dynamics of a two-qubit system coupled to a bath of harmonic oscillators. Using a partial \(P\) representation we analyzed the exact total state of an appropriate dephasing model. We derive conditions that enable us, for a large range of parameters, to detect whether the two-qubit–environment state is separable or entangled. Different coupling strengths and two-qubit initial states are considered.

The total system displays an interesting behavior when we look at the entanglement relation between the parts. Entanglement between the two-qubit system \(AB\) and the environment \(E\) will appear for low enough temperatures. For temperatures above a critical temperature, the total state remains separable. For most temperatures, and during relevant times for decoherence and loss of entanglement in the two-qubit state, the total state remains separable. Additionally, we show that the modes of the environment never get entangled. Therefore, for most parameters, the initial entanglement between the two qubits vanishes without any build-up of entanglement in any other bipartition. At very low temperatures we detect conditions where the total state oscillates between separable and entangled domains as a function of time. Remarkably, these oscillations can also be seen in the time evolution of the entanglement between the two qubits.

The investigation of SE separability and entanglement for the highly nontrivial total state, rests on a partial \(P\) representation. For entanglement detection we have to guess a good test state. This approach leads to “white” regions in the \((T,t)\) diagram, where we cannot make any statement about whether the total state is entangled or separable. Moreover, our approach allows one to detect SE entanglement, but we do not quantify it. Consequently, do not look at monogamy relations in the tripartite \(ABE\) system. A search for a way to estimate the amount of entanglement and verify these relations is still an open issue.

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