Studying partial hyperbolicity inside regimes of motion in Hamiltonian systems

Miguel A. Prado Reynoso, Rafael M. da Silva, Marcus W. Beims*
Departamento de Física, Universidade Federal do Paraná, Curitiba 81531-980, PR, Brazil

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A chaotic trajectory in weakly chaotic higher-dimensional Hamiltonian systems may locally present distinct regimes of motion, namely, chaotic, semidetermined, or ordered. Such regimes, which are consequences of dynamical traps, are defined by the values of the Finite-Time Lyapunov Exponents (FTLEs) calculated during specific time windows. The Covariant Lyapunov Vectors (CLVs) contain the information about the local geometrical structure of the manifolds, and the distribution of the angles between them has been used to quantify deviations from hyperbolicity. In this work, we propose to study the deviation of partial hyperbolicity using the distribution of the local and mean angles during each of the mentioned regimes of motion. A system composed of two coupled standard maps and the Hénon–Heiles system are used as examples. Both are paradigmatic models to study the dynamics of mixed phase-space of conservative systems in discrete and continuous dynamical systems, respectively. Hyperbolic orthogonality is a general tendency in strong chaotic regimes. However, this is not true anymore for weakly chaotic systems and we must look separately at the regimes of motion. Furthermore, the distribution of angles between the manifolds in a given regime of motion allows us to obtain geometrical information about manifold structures in the tangent spaces. The description proposed here helps to explain important characteristics between invariant manifolds that occur inside the regimes of motion and furnishes a kind of visualization tool to perceive what happens in phase and tangent space of dynamical systems. This is crucial for higher-dimensional systems and to discuss distinct degrees of (non)hyperbolicity.

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1. Introduction

For 2-dimensional weakly chaotic Hamiltonian systems, the chaotic motion and regular motion (or torus) are well separated in phase space, leading to a mixed dynamics. In higher-dimensional Hamiltonian systems with mixed phase-space, the regular and chaotic components are not separated, and the invariant tori are not isolated anymore [1]. In higher dimensions, the chaotic trajectory visits regions of the phase space ergodically and may suffer a dynamical trapping (or sticky motion) [2] close to the N-dimensional tori. While the mechanism of stickiness in 2-dimensional problems is well understood [3], in higher-dimensional systems, the mechanism which generates the sticky motion is still unclear [4–6]. Such kind of motion that typically occurs in Hamiltonian systems with mixed phase-space also affects the transport of particles [7,8]. Using the spectrum of Finite-Time Lyapunov Exponents (FTLEs) [9–11], some time ago the effect of the sticky motion on the chaotic trajectory was classified in distinct regimes of motion [12]. When all FTLEs are zero inside a time window, we have an ordered regime, when all are positive, the regime is chaotic. In between, we have regimes that are named semidetermined [12], even though it is possible to have positive FTLEs. While the concept of the regimes is well known, it was only recently that this concept was applied to improve the understanding of the dynamics in higher-dimensional weakly chaotic Hamiltonian systems: as a filtering procedure to the substantial increase in the characterization of the sticky motion [13], and to describe intermittent stickiness synchronization [14].

Besides the complex properties found in the dynamics of higher-dimensional systems, the main difficulty in describing such dynamics lies in our inability to visualize the motion in more than three dimensions properly. Thus, we need to find additional properties that help us in this task. In this sense, methods that allow the determination of the correct direction of stable, unstable, and central invariant manifolds in the tangent space of the dynamical systems are most welcome [15–19]. The mentioned directions, obtained from the Covariant Lyapunov Vectors (CLVs), not only allow us to calculate the angles between the invariant manifolds...
and enable a better visualization of what occurs in such higher-dimensional systems, but give also deeper insights about the origin of sticky motion in such systems. There have been many applications of CLVs to characterize the dynamics of complex systems [20-28], to mention a few. Of special interest is the use of CLVs to discuss distinct degrees of (non)hyperbolicity in dynamical systems [15,17,22,28,30].

The current work uses the angles between CLVs $\theta_i$ and the principle angles $\phi_i$ (angles between subspaces) to understand the complex dynamical structures in the mixed phase-space of weakly chaotic Hamiltonian systems. For this, we study the angles distribution in the ordered, semiordered, and chaotic regimes, mentioned before, for two systems. The first one is composed of two coupled standard maps, while the second consists of the continuous Hénon-Heiles system. Firstly, we show that for both systems, the regimes of motions can be attained by a proper choice of the time windows for the calculation of the FTEs. The relation of the regimes of motion and the angles is discussed below. The distribution of the angles $\theta_i$ and $\phi_i$ demonstrates that the regimes of motion are closely related to a specific behavior of such angles and are associated with distinct degrees of (non)hyperbolicity inside the regimes. The strategy adopted here allows a deeper understanding of the sticky motion in higher-dimensional Hamiltonian systems.

The plan of this paper is presented as follows. Section 2 describes the methods utilized in the numerical simulations, namely the CLVs, the angles $\theta_i$ between them, the principal angles $\phi_i$ used for the coupled maps, and the definition of regimes of motion. Section 3 discusses our numerical results. While Section 3.1 discusses the case of the coupled maps, Section 3.2 handles the continuous case of the Hénon-Heiles problem. For both examples, the regimes of motion are detected, and the distributions of the angles between CLVs are used to demonstrate our main results. Section 4 summarizes our findings.

2. Methods

2.1. Covariant Lyapunov vectors

Oseledets’s theorem [31-34] provides the necessary conditions for the existence of the decomposition of the tangent bundle called Oseledets’s splitting. Let $f : M \rightarrow M$ be an invertible measure-preserving transformation of a compact manifold of dimension $N$. It is possible to split the tangent space $TM$ for almost every point $x \in M$. In such way that

$$T_xM = E_x^u \oplus \cdots \oplus E_x^s,$$

(1)

with $dim(E_x^i) = n_i$. The subspaces $E_x^i$, called Oseledets subspaces, are invariant under the evolution $DF$ in the tangent bundle, that is $DF(E_x^i) = E_{f(x)}^i$. For all non zero $u \in E_x^i$, we have

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log \|DF^m(u)\| = \lambda_i.$$

The numbers $\lambda_i$ are called the asymptotic Lyapunov spectrum and $\eta_i$ correspond to their degeneracy. There are real numbers $\lambda_1 > \cdots > \lambda_N$ that $\eta_1 + \cdots + \eta_N = N$, where $k \leq n$. and positive numbers $n_1, \ldots, n_k \in \mathbb{Z}^+$. With the Oseledets’s splitting at point $x$, we can construct the following decomposition

$$T_xM = E_x^u \oplus E_x^s \oplus E_x^c,$$

(2)

where the central subspace, $E_x^c$, is associated with the asymptotic zero Lyapunov exponent, the unstable subspace, $E_x^u$, with the positive exponent and the stable subspace, $E_x^s$, with the negative exponent.

Obtaining a set of vectors that generates the Oseledets’s subspace allows us to scrutinize the directions of expansion/contraction of the phase space of a given system. Oseledets’s theorem naturally considers the CLVs $\nu_{i,x}$ as the unit vectors that generate the Oseledets subspaces $E_x^i$, that is, vectors independent of the norm, and invariant under temporal inversion, which are associated with the Lyapunov Exponents (LEs). The knowledge of the CLVs allows us to identify topological information about the local structure of the tangent space.

Although there are other algorithms to calculate the CLVs, for example, through the intersection of appropriate subspaces [24,35], in this work, we used the most popular method schematized by Ginelli et al. [38,19]. In this scheme, CLVs are calculated using a stable dynamic rule, based on the information obtained in the forward and backward evolution. In the case of non-degenerated values of $\lambda_i$, we can define (except for one sign) the CLVs $\nu_{i,x}$ associated with the corresponding LEs. In the case of degenerate LEs, any non-singular basis formed by $n_i = \dim E_x^i$ CLVs $\nu_{i,x}$ can be arbitrarily considered, with $i = 1, \ldots, n_i$.

2.2. Angles between the CLVs and hyperbolicity

The angles between the different Oseledets subspaces $E_x^i$ at the point $x$ can be used to characterize the dynamical properties of a chaotic system. They have been used as a direct method to detect whether or not a chaotic system is hyperbolic [15,16,36]. Furthermore, deviations from hyperbolicity can be studied by observing the behavior of the angles distributions when parameters are changed or even when compared to angles distributions from other systems [18,25,27,37]. The angle $\theta_{i,x}$ between the CLVs $\nu_{i,x}$ and $\nu_{j,x}$ at each point $x$ is defined by the inner product

$$\langle \nu_{i,x}, \nu_{j,x} \rangle = \cos(\theta_{i,j,x}),$$

$$\theta_{i,j,x} \in (0, \pi).$$

(3)

The CLVs corresponding to non-degenerate LEs cannot be completely parallel along a given trajectory. However, a trajectory can pass arbitrarily close to such tangencies, resulting in very small angles. Therefore, relevant physical information has to be encoded in the way that the probability distributions of the angles approach the zero angle. For simplicity, we will drop the subscript $x$ and use the notation $\theta_{ij}$ for the CLVs, and $\theta_{ij}$ for the local angles between them.

We can also define the mean value of the local angles $\theta_{ij}$, calculated over a trajectory of finite-time length $\omega$ ($\omega$ is a time window), which is represented by $\theta_{ij}^{(\omega)}$. Let $\nu_i$ and $\nu_j$ be the vectors associated with the invariant subspaces. If we discretize the trajectory in $m$ sections of length $\Delta t$, such that $m\Delta t = \omega$, then the mean value of the local angles can be defined by

$$\theta_{ij}^{(\omega)} = \frac{1}{\omega} \int_0^\omega dt \theta_{ij}(t) = \frac{1}{m\Delta t} \sum_{t=1}^{m} \theta_{ij},$$

(4)

where $\theta_{ij}$ is the local angle given by Eq. (3) along the considered trajectory. This quantity allows us to obtain insights about the most recurrent relative positions between the stable/unstable manifolds in a specific regime of motions in the trajectory during the time window $\omega$. We can also obtain information on the partial hyperbolicity deviation from the variance of the mean angles distributions. We say that a regime of motion has weak hyperbolicity if the probability of the angle between the unstable and stable subspace is close to zero, whereas has strong hyperbolicity when it is close to $\pi/2$.

2.3. Principal angles for the coupled maps case

In cases where the stable and unstable manifolds have a dimension larger than one, it is not sufficient to compute all the angles between pairs of vectors taken from two bases spanning the two subspaces. Instead, we need to consider angles between arbitrary linear combinations of such vectors [17]. These angles are called principal and are invariant under isometric transformation of the
Euclidean space. They can be obtained using singular value decomposition and will be applied here for the case of the coupled maps. In this section, we define the orthonormal basis used.

As before, let the set \( v_i, i = 1, \ldots, 4, \) be the CVs which generate the Oseledec’s subspaces, namely the unstable \( E^{u} = \text{span}(v_1, v_2) \) and stable \( E^{s} = \text{span}(v_3, v_4) \) subspaces. To obtain the principal angles we need an orthonormal basis to each of these subspaces. For this we choose \( E^{u} = \text{span}(f_1, f_2) \), so that

\[
f_1 = \frac{v_1 + v_2}{\|v_1 + v_2\|}, \quad f_2 = \frac{v_1 - v_2}{\|v_1 - v_2\|},
\]

for the unstable subspace and \( E^{s} = \text{span}(g_1, g_2) \), such that

\[
g_1 = \frac{v_3 + v_4}{\|v_3 + v_4\|}, \quad g_2 = \frac{v_3 - v_4}{\|v_3 - v_4\|},
\]

for the stable subspace. As described in details in Appendix A, the principal angles are obtained from

\[
\cos \phi_{1,2} = \sqrt{\lambda_{1,2}}, \quad \phi_{1,2} \in [0, \pi/2],
\]

with \( \lambda_{1,2} \) given by Eq. (17). Finally, \( \phi_1 \) and \( \phi_2 \) are the principal angles between stable and unstable manifolds in ascending order.

The angles \( \phi_{1,2} \) of the principle angles \( \phi_{1,2} \) are obtained by

\[
\phi_{1,2} = \frac{1}{\omega} \int_0^\omega dt \phi_{1,2}(t) = \frac{1}{m \Delta t} \sum_{i=1}^m \phi_{1,2,i},
\]

2.4. Definition of regimes

Now we will define the Lyapunov regimes, for which we need to compute the FTLEs spectrum \( \lambda_{1,2}^{(0)} \), calculated along a trajectory of length \( \omega \). Different from the asymptotic LFs mentioned in Section 2.1, the FTLEs are computed during a time window of size \( \omega \), with \( \lambda_{1}^{(0)} > \lambda_{2}^{(0)} > \ldots > \lambda_{N}^{(0)} > 0 \). The determination of the FTLEs allows us to explore temporal properties in the full spectrum of \( \lambda_{i}^{(0)} \) [13,14].

Consider a system with \( N \) degrees of freedom, a given trajectory belongs to a regime of type \( S_d \) if it has \( M \) FTLEs \( \lambda_{i}^{(0)} > \xi_l \), where \( \xi_l < \lambda_{i}^{(0)} \) are small thresholds. While regimes \( S_l \) with \( 0 < l < N \) are called semiordered, regimes \( S_0 \) and \( S_N \) are named ordered and chaotic, respectively. To choose the time window size \( \omega \) that undesirably separates the regimes of motion depends on the considered system and is the most difficult and delicate part of the numerical analysis. Many simulations are necessary to find the most satisfactory window, which must be large enough to have a trustful estimation of the FTLEs but also small enough to assure an acceptable resolution of the temporal variation of the \( \lambda_{i}^{(0)} \)'s. The computation of the FTLEs, which on average are in decreasing order, is obtained using the traditional Benettis algorithm [38,39]. For some time \( \tau \), inversions of the order \( \lambda_{i}^{(0)} > \lambda_{j}^{(0)} \) may occur and we choose to reorder the \( \lambda_{i}^{(0)} \) for all \( t \).

3. Numerical results

To present our results, we consider two systems, the coupled-map model representing discrete time systems and the Hénon–Heiles continuous dynamical system.

3.1. The coupled-maps model

A 2N-dimensional Hamiltonian system can be constituted by the time-discrete composition \( T = M \) of independent one-step iteration of \( N \) symplectic 2-dimensional maps \( M = (M_1, \ldots, M_N) \) and a symplectic coupling \( T = (T_1, \ldots, T_N) \), given by [40]

\[
M^{(i)}(p, x) = \begin{pmatrix}
p_i + K_i \sin(2\pi x_i) & \text{mod 1} \\
x_i + p_i + K_i \sin(2\pi x_i) & \text{mod 1}
\end{pmatrix},
\]

and for the coupling

\[
T^{(i)}(p, x) = \begin{pmatrix}
p_i + \sum_{j=1}^N K_{ij} \sin(2\pi x_j - x_i) & \text{mod 1}
\end{pmatrix},
\]

with \( K_{ij} = \delta_{ij} = e^{\frac{i \pi}{N-1}} \) being the all-to-all coupling strength. Variables \( (x_i, p_i) \) are pair-wise conjugated. In the numerical simulations, we used only the case \( N = 2 \), resulting in the 4-dimensional map proposed originally by Froeschlé [41,42], and used also in Ref. [43]. The nonlinear parameters corresponding to a mixed phase-space dynamics for the uncoupled case, namely \( K_1 = 0.54, K_2 = 0.55 \). For this case, the phase space has a large regular island surrounded by the chaotic sea. The coupled strength between the maps 1 and 2 is \( \xi = 0.01 \), and the coupled system has two positive FTLEs related to unstable manifolds and two negative FTLEs, related to stable manifolds.

3.1.1. Regimes and phase-space dynamics

Fig. 1 (a) displays the phase-space dynamics in the plane \( (x_1, p_1) \) for the decoupled standard map with \( K_1 = 0.54 \). A large integrability island is completely separated from the chaotic dynamics. When the maps are coupled, this island can be penetrated. In Fig. 1(b) and (c), the phase-space dynamics for the coupled system (9) and (10) is shown, with colors indicating respectively the angles \( \phi_1 \) and \( \phi_2 \) (see color bar). We observe in Fig. 1(b) that outside the island, which remains from the uncoupled case, chaotic trajectory has a mixture of angle between 0 and \( \pi/2 \). Fewer stripes with angles close to 0 are observed. Inside the island from the uncoupled case the angle \( \phi_1 \) is almost close to zero. Fig. 1(c) shows that outside the remaining island the angle \( \phi_2 \) varies between \( \sim 0 \) and \( \pi/2 \). Outside the remaining island, we observe many stripes with angles close to \( \pi/2 \), together with a mixture of angles \( \sim 0 \) and \( \pi/2 \). Therefore, while inside the remaining island the angle \( \phi_2 \approx 0 \), the angle \( 0 \approx \phi_2 \approx \pi/2 \), which suggests that the angle \( \phi_1 \) can detect tangencies between subspaces which occur inside approximated regular islands.

Results presented in Fig. 1 show that the angles \( \phi_1 \) and \( \phi_2 \) may strongly vary along the time evolution of the chaotic trajectories. These strong variations allow us to define the regimes of motion, as can be seen in Fig. 2(a), which displays a window of the time series. Plotted are the two largest FTLEs \( \lambda_1^{(0)} \) and \( \lambda_2^{(0)} \) as a function of time. The FTLEs are calculated inside a time window of \( \omega = 100 \) iterations. Some exemplary regimes of motion are marked by arrows in this figure, which are: the regime \( S_1 \), for which both FTLEs are larger than a threshold, \( \xi = 0.1 \) (dash-dotted line) for \( \lambda_1^{(0)} \) and \( \xi = 0.05 \) (dashed line) for \( \lambda_2^{(0)} \); the regime \( S_1 \), for which only \( \lambda_1^{(0)} \) is larger than \( \xi \); and the regime \( S_0 \), for which both FTLEs are smaller than their thresholds. For later purpose, we mention that Fig. 2(a) presents more regimes \( S_1 \), \( S_2 \), and \( S_2 \), not just those marked by the arrows. We already know that regimes \( S_1 \) and \( S_2 \) are associated with dynamical trappings that occur close to almost-invariant structures that live in high-dimensional phase spaces of conservative systems with mixed dynamics [13,44].

Our goal next is to understand what happens to the angles \( \phi_1 \) and \( \phi_2 \) in such regimes. Fig. 2(b) and (c) display the angles \( \phi_1 \) and \( \phi_2 \) in the same time window, respectively. For the regimes \( S_2 \) we observe that the two angles allow values between 0 and \( \pi/2 \), with a larger probability to have \( \phi_1 \) away from \( \pi/2 \) and \( \phi_2 \) away from 0. For the regimes \( S_1 \) we see that \( \phi_1 \) is strongly localized around 0, while the angle \( \phi_2 \) remains densely distributed in \( [0, \pi/2] \). Finally, in the ordered regime \( S_0 \), both angles tend to approach 0. However, \( \phi_2 \) never reaches zero, as will be confirmed later.

3.1.2. Regimes and distributions of \( \phi_1 \) and \( \phi_2 \)

The tendencies of the angles \( \phi_i (i = 1, 2) \) observed in the last section are restricted to the time window of the chaotic trajectory
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Their mean values are, however, away from 0 and π/2, presenting peaks located at φ₁(100) = 0.602 and φ₂(100) = 1.107.

Now we analyze the distributions obtained by separating the trajectory in regimes of motion. We observe in Fig. 3(b) that the main contributions to the tangencies, for which φ₁ = 0, comes from S₀ and S₁, in this order. This is confirmed by φ₁(100) in Fig. 3(e). The mean angles have maxima for φ₁(100) = 0.127 in the ordered regime S₀, φ₁(100) = 0.300 in the semiordered regime S₁ and φ₁(100) = 0.608 in S₂. Finally, we discuss the distribution of the angle φ₂, shown in Fig. 3(c). Clearly the angle φ₂ tends to zero for S₀ but never reaches it. On the other hand, in all regimes of motion the angle π/2 is observed. The mean angles have maxima for φ₂(100) = 0.497 in the ordered regime S₀, φ₂(100) = 0.940 in the semiordered regime S₁ and φ₂(100) = 1.109 in S₂.

Summarizing, the tangencies observed for φ₁ lead us to conclude that no hyperbolic orthogonality is allowed in any regime of motion.

3.2. Hénon–Heiles system

Due to the variety of dynamical properties in the phase space, the Hénon–Heiles system has become one of the simplest and most classic examples of continuous systems presenting chaotic dynamics. Proposed to describe the behavior of galaxies [45–47], the Hénon–Heiles system can also be seen as two oscillators coupled by a cubic perturbation. In this case, the Hamiltonian is given by

$$ H = \frac{p_x^2 + p_y^2}{2m} + \frac{m \omega^2}{2} (x_1^2 + y_1^2) + k_f (x_1^2 y_1 - \frac{y_1^3}{3}) $$

where m has mass unit, \( \omega > 0 \) has frequency unit [s⁻¹], and \( k_f \in \mathbb{R} \), being the coupling strength, has unit of [Kg/m²]. The total energy \( E = H \) is conserved. Using the adimensional variables \( p_x = p_{x,1}/m \omega \) and \( x = x_1/L \), where L has unit of length, the same goes for the variable y, the Hamiltonian (11) becomes the dimensionless Hamiltonian

$$ H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + x^2 y - \frac{y^3}{3}, $$

where the energy \( E = H \) is measured in units of \([H]/(m \omega^2 L^2)\) and \( k_f L/m \omega^2 = 1 \). As a consequence, the only parameter is the total energy of the system E. We denote \( \psi(E) \) to the flow generated by (12). Through the mapping in the Poincaré section defined at \( x = 0 \), it is well known that the trajectories are limited to a flat region inside the energy interval \( E = [0, 1/6] \). The growth of the chaotic region in the phase space is observed with increasing values of the

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energy. Here we will focus on the chaotic dynamics given by the energy $E = 1/6$.

Applying the Oseledec’s theorem to a point $x$ of a chaotic trajectory of the Hénon–Heiles system, the decomposition of Eq. (2) is valid with $\dim(E_{k,\perp}) = 1$ being the dimension of the unstable subspace, and $\dim(E_{k,\perp}) = 2$ being the dimension of the central subspace, and $\dim(E_{k,\parallel}) = 1$ being the dimension of the stable subspace. While the unstable and stable subspaces are generated by the CLVs $v_{1,k}$ and $v_{2,k}$, respectively, $E_{u,k} = \text{span}[v_{1,k}]$, $E_{c,k} = \text{span}[v_{1,k}, v_{2,k}]$, and $E_{s,k} = \text{span}[v_{2,k}]$. (13)

The central subspace is generated by $v_{2,k}$ and $v_{3,k}$ $E_{c,k} = \text{span}[v_{2,k}, v_{3,k}]$. (14)

The asymptotic LE spectrum is given by $\{\lambda, 0, 0, -\lambda\}$, with $\lambda \sim 0.126 > 0$ for $E = 1/6$. As a first study, we used the basis generated by the Ginelli’s algorithm for spanning these subspaces. However, for a system like Hénon–Heiles, another basis can be considered for spanning the central subspace, namely, the basis given by the direction along the tangent to the trajectory and in the direction of the gradient of the energy.

To integrate the Hénon–Heiles system, as well as to describe the evolution in the tangent space $D\theta^f(x)$, the fourth-order Runge-Kutta algorithm was used, with a time step $\Delta t = 0.01$. A time window $\omega = 250$ was used to compute the spectrum of FTLEs $\lambda_1^{(250)}$, with $i = 1, \ldots, 4$. Due to the symmetry of the LE spectrum, we consider only the first two FTLEs in our analysis. The simulations show that $\lambda_2^{(250)}$, associated with the central subspace, varies a small amount around zero. The value of the FTLE $\lambda_2^{(250)}$ allows us to define the chosen basis. In the case of the basis given by the Ginelli’s algorithm, our simulations show that $\theta_{12} \approx \theta_{13}$, and $\theta_{34} \approx \theta_{34}$, while $\theta_{23} \approx (0, \pi)$. Therefore, all relevant information about the dynamics in the tangent space can be given by the local angles $\theta_{12}$, $\theta_{34}$, and their respective mean values $\theta_{12}^{(250)}$, $\theta_{34}^{(250)}$, and $\theta_{23}^{(250)}$.

### 3.2.1. Regimes and phase-space dynamics

Fig. 4 presents the Poincaré surfaces of section (PSS) $(y, p_y)$ for the Hénon–Heiles system with colors indicating different physical quantities. In Fig. 4(a), the FTLE $\lambda_2^{(250)}$ is shown, calculated inside the window of length $\omega = 250$. The values of $\lambda_2^{(250)}$ change between 0 and 0.20, according to the color bar. For the energy $E = 1/6$, we have a mixed dynamics. We started initial conditions inside the chaotic region of the phase space so that the regular tori around the stable fixed points cannot be penetrated. The values of $\lambda_2^{(250)}$ in the chaotic region are large, while close to the regular tori, they decrease, as expected due to the sticky motion. Fig. 4(b)-(d) show, respectively, the same PSS but the angles $\theta_{12}$, $\theta_{13}$, and $\theta_{34}$ are denoted by the colors. It calls to attention the complexity in the possible values of the angles between 0 and $\pi$. From these Figures, no apparent distinction occurs close to the stable tori.

Many simulations were required to find the appropriate time window for $\lambda_1^{(250)}$, which undeniably separates the regimes of motion. To show this in more detail, we present in Fig. 5(a) the distribution of the FTLE but using colors to separate the threshold $\epsilon_1 = 0.02$. For values $\lambda_1^{(250)} < 0.02$ we used the black color, while values $\lambda_1^{(250)} > 0.02$ are indicated by the orange color. Fig. 5(b) displays the corresponding values in the PSS. This shows that when the chaotic trajectory is close to the regular tori, the FTLE $\lambda_2^{(250)}$ is bellow the threshold. Thus, the chosen threshold separates the motions affected by the tori.
Our first task is to check if the chosen threshold can adequately separate regimes of motion. In the Hénon–Heiles system, we have just one positive FTLE, and therefore only the chaotic regime $S_1$ and the ordered regime $S_0$ are allowed. This case, different from the coupled maps given by Eqs. (9) and (10), we do not have semiorbited regimes. Fig. 6(a) displays $\lambda_1^{(250)}$ inside a time interval showing clearly how both regimes can be separated. The threshold $\epsilon_1 = 0.02$ is indicated by the black dash-dotted line. Fig. 6(b)–(d) display, respectively, the time evolution of the angles $\theta_{12}$, $\theta_{14}$, and $\theta_{34}$, together with their mean value in a time window of $\omega = 250$. We notice that in the chaotic regime $S_1$, all angles are uniformly distributed between 0 and $\pi$, having a mean value $= \pi/2$. In the ordered regime $S_0$, the mean value of the angles starts to oscillate strongly, and values close to $\pi/2$ become rare. Especially the angle $\theta_{34}$ now approach of values 0 and $\pi$, meaning a near alignment. As in the coupled map case, also here the specific properties of angles between manifolds are responsible for the distinct regimes of motion.

### 3.2.2. Regimes and distributions of $\theta_{ij}$

As in the coupled map case, the tendencies of the angles, observed in Fig. 6(b)–(d), must be checked investigating the distribution of the angles inside the distinct regimes of motion. Results are shown in Fig. 7(a)–(d) for the local angles $\theta_{ij}$ and in Fig. 7(e)–(h) for their mean values. Fig. 7(a) and (e) show, respectively, the values of $\theta_{12}$, and their mean value, for the full trajectory. While the local values of $\theta_{12}$ tend to spread over the whole angle interval (except 0 and $\pi$), the distributions of the mean values $\theta_{12}$ look like Gaussian distributions, as expected for a chaotic system. All other figures display the angles distributions inside each regime of mo-

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**Fig. 5.** (Color online) (a) Distribution of the FTLE $\lambda_1^{(250)}$ and (b) the FSS (same as Fig. 4(a)) with black color for $\lambda_1^{(250)} > 0.02$ and orange color for $\lambda_1^{(250)} \geq 0.02$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 6.** (Color online) (a) Time series of the FTLEs $\lambda_1^{(250i)}$ ($i = 1, 2$) for the Hénon–Heiles system with energy $E = 1/6$, showing the regimes of motion $S_0$ and $S_1$. The threshold $\epsilon_1 = 0.02$ is represented by the black dash-dotted line, while the black dashed line indicates the value 0. The panels (b), (c) and (d) display, respectively, the time series of the angles $\theta_{12}$, $\theta_{14}$ and $\theta_{34}$ (black), together with their mean values $\theta_{12}^{(250)}$, $\theta_{14}^{(250)}$ and $\theta_{34}^{(250)}$ (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 7.** (Color online) Distributions of the angles $\theta_{ij}$ [(a)–(d)] and of the mean values $\theta_{ij}^{(250)}$ [(e)–(h)] for the Hénon–Heiles system with $E = 1/6$. In (a) and (e), the full trajectory was considered. In (b) and (f) ($\theta_{12}$), (c) and (g) ($\theta_{14}$), and (d) and (h) ($\theta_{34}$), only angles inside each regime of motion were considered. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
tion. For the chaotic regime $S_1$, we observe, as for the full trajectory, that the local values of $\theta_1$ tend to spread over the whole angle interval (except 0 and $\pi$), while their mean values $\theta_1^{(k)}$ approach Gaussian distributions. On the other hand, the local angles $\theta_2$ of the ordered regime $S_2$ have large probabilities away from $\pi/2$ but also tend to zero close to 0 and $\pi$. Exception is the angle $\theta_3$, for which we found large probabilities at the borders, which means that a near alignment between the unstable and the stable mani-

4. Conclusions

Ordered, semiordered and chaotic regimes of motion were used recently to substantially increase the characterization of the sticky motion using recurrence plots [44] and to describe intermittent stickiness synchronization [44]. In the present work, we characterize the dynamical structure inside such regimes of motion using the angles between CLVs in the tangent space.

For this study, we used two weakly chaotic Hamiltonian systems, namely, the map composed of $N = 2$ coupled standard maps, and the Hénon-Heiles system, representing discrete and continuous problems, respectively. We summarize our results by presenting the main properties regarding the angles between CLVs found separately in each regime.

Chaotic regimes: We start discussing the discrete coupled maps case. The principal angles $\phi_1$ and $\phi_2$ tend to show that we do not have hyperbolic orthogonality. The corresponding mean values have distributions which are sharp at specific angles. For the Hénon-Heiles case, all mean angles $\theta_{1,2}^{(k)}$ have larger probability $P(\theta_{1,2}^{(k)})$ close to $\theta_{1,2}^{(k)} \sim \pi/2$. The local angles are distributed in the whole interval (0, $\pi$), but near tangencies between manifolds are not observed, which implies less frequent violations of hyperbolicity, but no strict hyperbolicity [18].

Semiordered regimes: Only the discrete maps case has such a regime, in this case it is $S_1$. Relative violation of hyperbolicity is observed only for $\phi_1$. The corresponding mean values have distributions which are sharp at specific angles.

Ordered regimes: For the discrete maps in the ordered regimes $S_0$, stronger non-hyperbolicity has observed for $\phi_1$, and the corresponding mean values have distributions which are sharp at specific angles.

Finally, the technique used in this work can be applied for weakly chaotic Hamiltonian systems with larger dimensions than those presented here. It allows for a geometrical understanding of the underlying dynamics in different regimes of motion. Thus, it is a promising way to combine numerical simulations and mathematical properties to understand more the complex properties found in the dynamics of higher-dimensional Hamiltonian systems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

From the orthonormal basis we construct the column matrices $Q_1 = [f_1, f_2]$ and $Q_2 = [g_1, g_2]$. The cosine of the principal angles, $\cos \phi_{1,2}$, are defined as the elements of the diagonal matrix in the Single Value Decomposition (SVD) factorization of the matrix $Q_1^T Q_2$, which is written as

$$Q_1^T Q_2 = \begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$$

The elements of this matrix can be expressed in terms of the CLVs as

$$Q_1^T Q_2 = \begin{bmatrix} \cos \theta_1 + \cos \theta_2 \cos \theta_3 + \cos \theta_4 \\ 2\sqrt{(1 + \cos \theta_2)(1 + \cos \theta_4)} \\ \cos \theta_1 - \cos \theta_2 \cos \theta_3 + \cos \theta_4 \\ 2\sqrt{(1 + \cos \theta_2)(1 - \cos \theta_4)} \end{bmatrix}$$

The following we denote $Q = Q_1^T Q_2$. By the relations between the SVD and the eigenvalue decomposition, the non-zero elements of the diagonal matrix in the SVD of $Q_1^T Q_2$ are the square roots of the non-zero eigenvalues of $Q_1^T Q_1$ or $Q_2^T Q_2$. Then, the cosines of the principal angles between $Q_1$ and $Q_2$ are the square roots of the eigenvalues of the $2 \times 2$ matrix $Q_1^T Q_2$. In this way we obtain

$$Q_1^T Q_2 = \begin{bmatrix} (f_1 | g_1) & (f_1 | g_2) \\ (f_2 | g_1) & (f_2 | g_2) \end{bmatrix}$$

so that the eigenvalues of the $2 \times 2$ matrix $Q_1^T Q_2$ are

$$\lambda_{1,2} = \frac{1}{2} \left[ T \pm \sqrt{T^2 - 4D} \right]$$

where $T = \text{Tr}(Q_1^T Q_2)$ is the trace of the matrix and $D = \text{Det}(Q_1^T Q_2)$ its determinant, given respectively by

$$T = (f_1 | g_1)^2 + (f_1 | g_2)^2 + (f_2 | g_1)^2 + (f_2 | g_2)^2,$$

$$D = \left( (f_1 | g_1)(f_2 | g_2) - (f_1 | g_2)(f_2 | g_1) \right)^2.$$
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